

SCATTERING MATRIX AND ANALYTIC TORSION

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ABSTRACT. For a compact manifold, which has a part isometric to a cylinder of finite length, we consider an adiabatic limit procedure, in which the length of the cylinder tends to infinity. We study the asymptotic of the spectrum of Hodge-Laplacian and the asymptotic of the L^2 -metric on de Rham cohomology. As an application, we give a pure analytic proof of the gluing formula for analytic torsion.

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0. INTRODUCTION

Given a manifold with cylindrical ends, that is a non-compact Riemannian manifold whose non-compact part is isometric to a cylinder, the spectrum of its Laplacian has an absolutely continuous part, which is determined by scattering matrix. Intuitively, scattering matrix encodes how an incoming wave on the cylinder is scattered by the compact part of the manifold.

Now, consider a compact Riemannian manifold which has a part isometric to a (finite length) cylinder and stretch the cylinder so that its length tends to infinity. This procedure is referred as taking the adiabatic limit. At least intuitively, this manifold converges to the disjoint union of two manifolds with cylindrical ends, which interact : a wave coming from one manifold is scattered by the other. This intuition guides us to describe the adiabatic behavior of the spectrum of Laplacian using scattering matrix.

This adiabatic limit setting first appeared in the work of Douglas-Wojciechowski [16] for studying η -invariant. Müller [31] studied the η -invariant of manifolds with cylindrical ends using scattering theory. Park-Wojciechowski [33] studied the adiabatic behavior of the spectrum of Dirac operator using scattering matrix. Their result is a refinement of earlier works of Cappell-Lee-Miller [14]. We remark that the adiabatic limit mentioned above is a variation of the standard adiabatic limit ([4, 5], etc.), which is associated with a fibration.

In this paper, we study the adiabatic properties of Hodge-de Rham operators. We pay particular attention to those properties which do not hold for general Dirac operators. The scattering matrix plays a central role in our study.

One of the main ingredients in this paper is an asymptotic estimate of the spectrum of Hodge-de Rham operator under adiabatic limit. As a consequence, we give an asymptotic gluing formula for the ζ -determinant of Hodge-Laplacian.

Another main ingredient in this paper is an asymptotic estimate of the L^2 -metric on de Rham cohomology under adiabatic limit. Rather than considering a single manifold, we work with a gluing setting. Then the cohomologies in question fit into a Mayer-Vietoris sequence. As a consequence, we calculate the adiabatic limit of the torsion associated with the Mayer-Vietoris sequence. As an application of our results, we give a new proof of the gluing formula for analytic torsion.

Now, we explain more about analytic torsion. Given a flat vector bundle F equipped with metric on a compact Riemannian manifold Z , the Ray-Singer analytic torsion [34] is a (weighted) product of the determinants of the Hodge Laplacian twisted by F , and the Ray-Singer metric on the determinant of $H^\bullet(Z, F)$ is the product of its L^2 -metric and the Ray-Singer analytic torsion. The analytic torsion has a topological

counterpart, known as the Reidemeister torsion [36]. Ray and Singer [34] conjectured that the two torsions coincide. For unitarily flat vector bundles, this conjecture is proved independently by Cheeger [15] and Müller [29]. Bismut-Zhang [10] and Müller [30] simultaneously considered generalizations of this result. Müller [30] extended this result to the case where the dimension of the manifold is odd and only the metric induced on $\det F$ is required to be flat. Bismut-Zhang [10] generalized this result to arbitrary flat vector bundles with arbitrary Hermitian metrics. There are also various extensions to the equivariant case [25, 26, 11].

The gluing formula for analytic torsion considered in this paper is the following : when one has a hypersurface $Y \subseteq Z$ cutting Z into two submanifolds $Z = Z_1 \cup_Y Z_2$, is there an additive formula linking three analytic torsions (for Z_1 , Z_2 and Z) ? This problem was first formulated by Ray-Singer [34] as a possible approach to Ray-Singer conjecture. It is proved for unitary flat vector bundles with product structure metrics near Y by Lück [26], Vishik [38], and proved in full generality by Brüning-Ma [13]. There are also related works of [19] and [24].

The family version of analytic torsion is constructed by Bismut-Lott [9] (BL-torsion). Under the hypothesis that there exists a fiberwise Morse function, Bismut-Goette [7] obtained a family version of the Bismut-Zhang theorem, i.e., a formula linking BL-torsion to higher Reidemeister torsion ([20, 17, 2], see also [18] for a survey). It is conjectured (conference on the higher torsion invariants, Göttingen, September 2003) that there should exist a gluing formula for BL-torsion. This conjecture may serve as an intermediate step to under the relation between BL-torsion and higher Reidemeister torsion in general, conjectured by Igusa [21]. Zhu [41] formulated the conjectured gluing formula and proved it under the same hypothesis as Bismut-Goette's [7].

The proof of the gluing formula for analytic torsion done in this paper is purely analytic. It is generalizable for BL-torsion. Our strategy is also used by Zhu [40] for proving the gluing formula for BL-torsion under the hypothesis that $H^\bullet(Y, F) = 0$. We remark that $H^\bullet(Y, F) = 0$ implies the absence of s-values (cf. §0.2) and the splitting of the Mayer-Vietoris sequence.

Let us now give more detail about the matter of this paper.

0.1. Manifolds with cylindrical ends and scattering matrices. Let X be a compact manifold with boundary $\partial X = Y$. We fix $U =]-1, 0] \times Y$ a collar neighborhood of ∂X . Let $\pi_Y :]-1, 0] \times Y \rightarrow Y$ be the natural projection. Let F be a flat complex vector bundle on X with flat connection ∇^F , i.e., $\nabla^{F,2} = 0$. Using parallel transport along $u \in]-1, 0]$, $(F|_U, \nabla^F|_U)$ is identified to $\pi_Y^*(F|_Y, \nabla^F|_Y)$ (cf. (2.7)).

We equip X with a Riemannian metric g^{TX} and F with a Hermitian metric h^F . Let g^{TY} be the metric on Y induced by g^{TX} . We suppose that (cf. [13, (2.1) and (2.3)])

$$(0.1) \quad g^{TX}|_U = du^2 + g^{TY}, \quad h^F|_U = \pi_Y^*(h^F|_Y) .$$

For $0 \leq R \leq \infty$, set $X_R = X \cup_Y [0, R] \times Y$ with $U_R := U \cup [0, R] \times Y =]-1, R] \times Y$ the cylindrical part of X_R . Still, let $\pi_Y :]-1, R] \times Y \rightarrow Y$ be the natural projection. Then, F extends to X_R in the natural way, i.e., $(F, \nabla^F)|_{U_R} = \pi_Y^*(F|_Y, \nabla^F|_Y)$. We extend equally g^{TX} and h^F to X_R , such that (0.1) holds with U replaced by U_R .

Let $\Omega^\bullet(X_R, F)$ be the vector space of differential forms on X_R with values in F . Let $d^F : \Omega^\bullet(X_R, F) \rightarrow \Omega^{\bullet+1}(X_R, F)$ be the de Rham operator induced by ∇^F , let $d^{F,*}$ be its

formal adjoint (with respect to L^2 -metric). Set

$$(0.2) \quad D_{X_R}^F = d^F + d^{F,*},$$

called Hodge-de Rham operator. Its square $D_{X_R}^{F,2}$ is called Hodge-Laplacian.

For $R = \infty$, the spectrum of $D_{X_\infty}^{F,2}$ has an absolutely continuous part (cf. [35, §7.2]).

Let $\mathcal{H}^\bullet(Y, F) \subseteq \Omega^\bullet(Y, F)$ be the kernel of D_Y^F , the Hodge-de Rham operator on Y . Set $\mathcal{H}^\bullet(Y, F[du]) = \mathcal{H}^\bullet(Y, F) \oplus \mathcal{H}^\bullet(Y, F)du$. We fix $\delta_Y > 0$, such that $] - \delta_Y, \delta_Y[\cap \text{Sp}(D_Y^F) \subseteq \{0\}$. The scattering matrix (cf. [22, Theorem 1], [31, §4])

$$(0.3) \quad C(\lambda) \in \text{End}(\mathcal{H}^\bullet(Y, F[du])) , \quad \lambda \in] - \delta_Y, \delta_Y[,$$

is characterized by the following property : for ω a generalized eigensection (cf. §2.3) of $D_{X_\infty}^F$ with eigenvalue $\lambda \in] - \delta_Y, \delta_Y[$, there exist $\phi \in \mathcal{H}^\bullet(Y, F[du])$ and $\theta \in \mathcal{C}^\infty([0, \infty[, \Omega^\bullet(Y, F[du]))$, which is L^2 -integrable, such that (cf. (2.31))

$$(0.4) \quad \omega|_{U_\infty} = e^{-i\lambda u} \phi + e^{i\lambda u} C(\lambda) \phi + \theta .$$

It is reasonable to expect that the asymptotic limit (as $R \rightarrow \infty$) of certain invariant of X_R can be expressed by certain data of X_∞ , in particular, by the scattering matrix.

0.2. Asymptotics of the spectrum of Hodge-Laplacian. Let (Z, g^{TZ}) be a closed Riemannian manifold. Let $Y \subseteq Z$ be a hypersurface cutting Z into two pieces, say Z_1 and Z_2 . Then $\partial Z_1 = \partial Z_2 = Y$ and $Z = Z_1 \cup_Y Z_2$. Let (F, ∇^F) be a flat complex vector bundle over Z . The restriction of F to Z_1 or Z_2 is still denoted F . Let h^F be a metric on F . We suppose that g^{TZ} and h^F have product structure near Y , in the sense of (0.1).

Following the same procedure in §0.1 to construct X_R from X , we construct Riemannian manifold $Z_{j,R}$ ($j = 1, 2$) from Z_j . For $R \in [0, \infty[$, set $Z_R = Z_{1,R} \cup_Y Z_{2,R}$. Then (F, ∇^F, h^F) extends to Z_R by respecting (2.7) and (0.1).

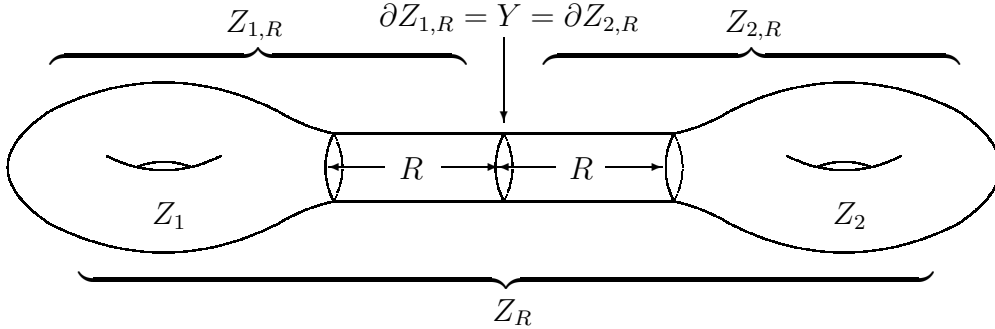


FIGURE 1.

In the whole paper, we will always use the relative boundary condition on $Z_{1,R}$ and the absolute boundary condition on $Z_{2,R}$ (cf. (1.5)). Let $D_{Z_R}^F$ be the Hodge-de Rham operator (cf. (0.2)) acting on $\Omega^\bullet(Z_R, F)$. We define equally $D_{Z_{j,R}}^F$ ($j = 1, 2$), the Hodge-de Rham operator acting on $\Omega_{\text{bd}}^\bullet(Z_{j,R}, F)$ (cf. (1.5)).

As $R \rightarrow \infty$, the behaviors of eigenvalues of $D_{Z_R}^F$ are classified by Cappell-Lee-Miller [14, Theorem A] into the following three types :

- large eigenvalues (l-values), which remains uniformly away from 0 ;
- polynomially small eigenvalues (s-values), which tend to zero with speed slower than $R^{-1-\varepsilon}$ for any $\varepsilon > 0$;
- exponentially small eigenvalues (e-values), which lie in $[-e^{-cR}, e^{-cR}]$ for certain $c > 0$ and there are only finitely many of them.

Using scattering matrix, Park-Wojciechowski [33, Theorem 3.5] give an estimate of the s-values lying in $[-R^{-\varepsilon}, R^{-\varepsilon}]$ with $\mathcal{O}(e^{-cR})$ error term. They also show that the e-values are identically zero under the hypothesis that $\nabla^F h^F = 0$ [33, Proposition 3.9].

In this paper, we obtain the following result (see Theorem 3.18) : for any Hodge-Rham operator, there exists $\delta > 0$, such that the estimate (3.142) holds for s-values lying in $[-\delta, \delta]$, and e-values are identically zero. We also extends our results to manifolds with boundaries equipped with relative/absolute boundary condition (see Theorem 4.7). As a consequence, we prove a gluing formula for ζ -determinant under adiabatic limit, which is stated in the sequel.

Let N be the number operator on $\Omega^\bullet(Z_R, F)$, i.e., for $\omega \in \Omega^p(Z_R, F)$, $N\omega = p\omega$. Let $P : \Omega^\bullet(Z_R, F) \rightarrow \ker(D_{Z_R}^{F,2})$ be the orthogonal project with respect to the L^2 -metric. The ζ -function associated with $D_{Z_R}^{F,2}$ is defined, for $s \in \mathbb{C}$ with $\text{Re}(s) > \frac{1}{2} \dim Z$, by

$$(0.5) \quad \zeta_R(s) = -\text{Tr} \left[(-1)^N N \left(D_{Z_R}^{F,2} \right)^{-s} (1 - P) \right] .$$

Then ζ_R admits a meromorphic extension to the whole complex plane \mathbb{C} , which is regular at $0 \in \mathbb{C}$. Let $\zeta_R^p(s)$ be (0.5) with $D_{Z_R}^{F,2}$ replaced by $D_{Z_R}^{F,2,(p)} := D_{Z_R}^{F,2}|_{\Omega^p(Z_R, F)}$. Then

$$(0.6) \quad \exp(\zeta_R'(0)) = \prod_{p=1}^{\dim Z} \left(\exp(\zeta_R^p(0)) \right)^p ,$$

i.e., it is a weighted product of ζ -determinants of $D_{Z_R}^{F,2,(p)}$. In the sequel, we call $\exp(\zeta_R'(0))$ the (weighted) ζ -determinant of $D_{Z_R}^{F,2}$. In the same way, we define $\zeta_{j,R}(s)$, the ζ -function associated with $D_{Z_{j,R}}^{F,2}$.

Let $C_j(\lambda) \in \text{End}(\mathcal{H}^\bullet(Y, F[du]))$ ($j = 1, 2, \lambda \in \mathbb{R}$) be the scattering matrix associated with $\Omega^\bullet(Z_{j,\infty}, F)$. For $p = 0, \dots, \dim Z$, we denote

$$(0.7) \quad C_{12} = (C_2^{-1} C_1)(0) , \quad C_{12}^p = C_{12}|_{\mathcal{H}^p(Y, F) \oplus \mathcal{H}^{p-1}(Y, F) du} .$$

Set

$$(0.8) \quad \begin{aligned} \chi'(C_{12}) &= \sum_{p=0}^{\dim Z} p(-1)^p \dim \ker(C_{12}^p - 1) , \\ \chi' &= \sum_{p=0}^{\dim Z} p(-1)^p \left\{ \dim H^p(Z, F) - \dim H_{\text{bd}}^p(Z_1, F) - \dim H_{\text{bd}}^p(Z_2, F) \right\} , \\ \chi(Y, F) &= \sum_{p=0}^{\dim Y} (-1)^p \dim H^p(Y, F) , \end{aligned}$$

where $H_{\text{bd}}^\bullet(\cdot, F)$ is defined by (0.23).

For any Hermitian matrix A , let $\det^*(A)$ be the product of its non zero eigenvalues.

Theorem 0.1. *For any $\varepsilon > 0$, as $R \rightarrow +\infty$, we have*

$$(0.9) \quad \zeta_R'(0) - \zeta_{1,R}'(0) - \zeta_{2,R}'(0) = 2\chi' \log R + (\chi(Y, F) + \chi'(C_{12})) \log 2 \\ + \sum_{p=0}^{\dim Z} \frac{p}{2} (-1)^p \log \det^* \left(\frac{2 - C_{12}^p - (C_{12}^p)^{-1}}{4} \right) + \mathcal{O}(R^{-1+\varepsilon}).$$

We remark that such asymptotic gluing formulas for ζ -determinants in similar contexts are studied by Müller-Müller [28] and Park-Wojciechowski [33].

0.3. Analytic torsion and Mayer-Vietoris sequence. For λ a complex line, let $\lambda^{-1} = \lambda^*$ be its dual. Let E be a finite dimensional real or complex vector space, its determinant line is defined as $\det E = \Lambda^{\max} E$. More generally, for a \mathbb{Z} -graded finite dimensional vector space $E^\bullet = \bigoplus_{k=0}^n E^k$, we define

$$(0.10) \quad \det E^\bullet = \bigotimes_{k=0}^n (\det E^k)^{(-1)^k}.$$

For

$$(0.11) \quad (V^\bullet, \partial) : 0 \rightarrow V^0 \rightarrow V^1 \rightarrow \dots \rightarrow V^n \rightarrow 0$$

an exact sequence of finite dimensional vector spaces, there is a canonical section $\varrho \in \det V^\bullet$: let $m_j = \dim \text{Im}(\partial|_{V^j})$, we choose $(s_{j,k})_{1 \leq k \leq m_j}$ in V^j such that they project to a basis of $V^j / \partial V^{j-1}$, then with $\wedge_k s_{j,k} := s_{j,1} \wedge \dots \wedge s_{j,m_j}$, we define

$$(0.12) \quad \varrho = \bigotimes_{j=0}^n \left((\wedge_k \partial s_{j-1,k}) \wedge (\wedge_k s_{j,k}) \right)^{(-1)^j} \in \det V^\bullet.$$

Let g^{V^\bullet} be metric on V^\bullet . Let ∂^* be the adjoint of ∂ . Then $(\partial + \partial^*)^2 = \partial\partial^* + \partial^*\partial$ preserves each V^j . The torsion (cf. [6, Definition 1.4]) of (V^\bullet, ∂) is defined by

$$(0.13) \quad \mathcal{T}(V^\bullet, \partial) = \prod_j [\det((\partial + \partial^*)^2|_{V^j})]^{(-1)^{j/2}} \in \mathbb{R}_+.$$

Let $\|\cdot\|_{\det V^\bullet}$ be the induced metric on $\det V^\bullet$, then (cf. [6, Proposition 1.5])

$$(0.14) \quad \mathcal{T}(V^\bullet, \partial) = \|\varrho\|_{\det V^\bullet}.$$

We recall that $Z_{1,R}$, $Z_{2,R}$, Z_R and F are defined in §0.2. We consider the following Mayer-Vietoris sequence

$$(0.15) \quad \dots \rightarrow H_{\text{bd}}^p(Z_{1,R}, F) \rightarrow H^p(Z_R, F) \rightarrow H_{\text{bd}}^p(Z_{2,R}, F) \rightarrow \dots,$$

which is equipped with L^2 -metrics. Let \mathcal{T}_R be its torsion.

Theorem 0.2. *As $R \rightarrow \infty$, we have*

$$(0.16) \quad \mathcal{T}_R = 2^{\chi'(C_{12})/2} R^{\chi'} \prod_{p=0}^{\dim Z} \det^* \left(\frac{2 - C_{12}^p - (C_{12}^p)^{-1}}{4} \right)^{\frac{p}{4}(-1)^p} + \mathcal{O}(R^{\chi'-1}).$$

Viewing the Mayer-Vietoris sequence (0.15) with $R = 0$ as an acyclic complex and applying (0.12), we get the canonical section

$$(0.17) \quad \varrho \in \lambda(F) := \left(\det H^\bullet(Z, F) \right)^{-1} \otimes \det H_{\text{bd}}^\bullet(Z_1, F) \otimes \det H_{\text{bd}}^\bullet(Z_2, F) .$$

We use the conventions that $Z_0 = Z$ and $H_{\text{bd}}^\bullet(Z_0, F) = H^\bullet(Z, F)$. Let $\zeta_j(s)$ ($j = 0, 1, 2$) be the ζ -functions (cf. (0.5)) associated with the Hodge Laplacian $D_{Z_j}^{F,2}$. Let $\|\cdot\|_{\det H_{\text{bd}}^\bullet(Z_j, F)}^{L^2}$ be the L^2 -metric on $\det H_{\text{bd}}^\bullet(Z_j, F)$.

The Ray-Singer metric on $\det H_{\text{bd}}^\bullet(Z_j, F)$ ($j = 0, 1, 2$) is defined by

$$(0.18) \quad \|\cdot\|_{\det H_{\text{bd}}^\bullet(Z_j, F)}^{\text{RS}} = \|\cdot\|_{\det H_{\text{bd}}^\bullet(Z_j, F)}^{L^2} \exp \left(\frac{1}{2} \zeta_j'(0) \right) .$$

Let $\|\cdot\|_{\lambda(F)}^{\text{RS}}$ be the product norm on $\lambda(F)$ induced by $\|\cdot\|_{\det H_{\text{bd}}^\bullet(Z_j, F)}^{\text{RS}}$. The following theorem is first proved by Brüning-Ma [13, Theorem 0.3].

Theorem 0.3. *If g^{TZ} and h^F have product structures near Y (cf. (0.1)), then*

$$(0.19) \quad \|\varrho\|_{\lambda(F)}^{\text{RS}} = 2^{-\frac{1}{2}\chi(Y, F)} .$$

Let $\mathcal{T} = \mathcal{T}_0$. Then (0.19) can be reformulated as follows.

$$(0.20) \quad \frac{1}{2} \zeta'(0) - \frac{1}{2} \zeta_1'(0) - \frac{1}{2} \zeta_2'(0) - \log \mathcal{T} = \frac{1}{2} \chi(Y, F) \log 2 .$$

In this paper, we give a direct proof of (0.20) : by Theorem 0.1, 0.2, we know that $t_R := \frac{1}{2} \zeta_R'(0) - \frac{1}{2} \zeta_{1,R}'(0) - \frac{1}{2} \zeta_{2,R}'(0) - \log \mathcal{T}_R$ tends to $\frac{1}{2} \chi(Y, F) \log 2$ as $R \rightarrow \infty$, meanwhile, using the anomaly formula for analytic torsion [10, Theorem 0.1], we know that t_R is independent to R . This proves (0.20).

This paper is organized as follows. In §1, we review some results concerning absolute/relative cohomology of manifolds with boundary and the Mayer-Vietoris sequence. In §2, we review some results on the spectrum of the Hodge Laplacian on a manifold with cylindrical ends and introduce scattering matrix. In §3, we study the spectrum of the Hodge Laplacian on the stretched manifold Z_R , and link it to scattering theory. In §4, we prove similar results for manifolds with boundary. In §5, we prove Theorem 0.1. In §6, we prove Theorem 0.2. In §7, we give our new proof of Theorem 0.3.

0.4. Notations. Hereby, we summarize some frequently used notations in this paper.

A manifold (with or without boundary) is usually denoted by X, Y or Z . We denote by g^{TX} a Riemannian metric on X . We always consider manifold equipped with a flat complex vector bundle F with flat connection ∇^F and a Hermitian metric h^F .

By $\Omega^\bullet(X, F)$, we mean the vector space of differential forms on X with values in F . We denote by $\Omega_c^\bullet(X, F)$ the subspace of forms that are compactly supported.

By $\|\cdot\|_X$, we mean the L^2 -metric on $\Omega^\bullet(X, F)$. More precisely, let $\langle \cdot, \cdot \rangle_{\Lambda^\bullet(T^*X) \otimes F}$ be the scalar product on $\Lambda^\bullet(T^*X) \otimes F$ induced by g^{TX} and h^F . Let dv_X be the Riemannian volume form on X , then, for $\omega \in \Omega^\bullet(X, F)$, we have

$$(0.21) \quad \|\omega\|_X^2 = \int_X \langle \omega_x, \omega_x \rangle_{\Lambda^\bullet(T^*X) \otimes F} dv_X(x) .$$

The scalar product associated with $\|\cdot\|_X$ is denoted by $\langle \cdot, \cdot \rangle_X$. By $\|\cdot\|_{\mathcal{C}^0, X}$, we mean the \mathcal{C}^0 -norm on $\Omega^\bullet(X, F)$. More precisely,

$$(0.22) \quad \|\omega\|_{\mathcal{C}^0, X}^2 = \sup \left\{ \langle \omega_x, \omega_x \rangle_{\Lambda^\bullet(T^*X) \otimes F} : x \in X \right\}.$$

By d^F , we mean the de Rham operator acting on $\Omega^\bullet(X, F)$ induced by ∇^F . By $d^{F,*}$, we mean the formal adjoint of d^F . The Hodge-de Rham operator is defined by $D_X^F = d^F + d^{F,*}$.

We denote

$$(0.23) \quad H_{\text{abs}}^\bullet(X, F) = H^\bullet(X, F), \quad H_{\text{rel}}^\bullet(X, F) = H^\bullet(X, \partial X, F).$$

We write $H_{\text{bd}}^\bullet(X, F)$ for short if the choice of abs/rel is clear.

By the L^2 -metric on $H_{\text{bd}}^\bullet(X, F)$, we mean the metric induced by $\|\cdot\|_X$ via Hodge theorem (cf. Theorem 1.1).

If A is self-adjoint operator, we denote by $\text{Sp}(A)$ its spectrum.

For any Hermitian matrix A , we note

$$(0.24) \quad \det^*(A) = \prod_{\lambda \in \text{Sp}(A) \setminus \{0\}} \lambda.$$

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1. COHOMOLOGIES FOR MANIFOLDS WITH BOUNDARY

In this section, we review some basic constructions and results about the cohomology of a compact manifold with boundary.

In §1.1, using the language of simplicial complex, we define the absolute/relative cohomology of a compact manifold with boundary with values in a flat vector bundle. In §1.2, we state the Hodge theory for absolute/relative cohomology. In §1.3, we state the classical Mayer-Vietoris sequence in simplicial cohomology together with its interpretation in de Rham cohomology and Hodge theory.

1.1. Absolute/Relative cohomology. Let X be a compact \mathcal{C}^∞ -manifold with boundary $\partial X = Y$. Let $F \rightarrow X$ be a flat complex vector bundle equipped with flat connection ∇^F . Let F^* be the dual vector bundle of F .

Let K_X be a smooth triangulation of X , such that $K_Y = K_X \cap Y$ gives a triangulation of Y . For $0 \leq p \leq \dim X$, let $K_X^p \subseteq K_X$ be the set of cells in K_X of dimension $\leq p$. Let B be the set of barycenters of the simplexes in K_X . Let $b : K_X \rightarrow B$ be the obvious

one-to-one map. If $\mathfrak{a} \in K_X$, let $[\mathfrak{a}]$ be the real line generated by \mathfrak{a} . Let $(C_\bullet(K_X, F^*), \partial)$ be the complex of simplicial chains in K_X with values in F^* . We have

$$(1.1) \quad C_p(K_X, F^*) = \bigoplus_{\mathfrak{a} \in K_X^p \setminus K_X^{p-1}} [\mathfrak{a}] \otimes_{\mathbb{R}} F_{b(\mathfrak{a})}^* .$$

The chain map ∂ maps $C_p(K_X, F^*)$ to $C_{p-1}(K_X, F^*)$. Then $(C_\bullet(K_Y, F^*), \partial)$ is a subcomplex of $(C_\bullet(K_X, F^*), \partial)$. We can then define the quotient complex $(C_\bullet(K_X/K_Y, F^*), \partial)$, such that

$$(1.2) \quad C_\bullet(K_X/K_Y, F^*) = C_\bullet(K_X, F^*) / C_\bullet(K_Y, F^*) .$$

If $\mathfrak{a} \in K_X$, let $[\mathfrak{a}]^*$ be the real line dual to $[\mathfrak{a}]$. Let $(C^\bullet(K_X, F), \tilde{\partial})$ be the complex dual to $(C_\bullet(K_X, F^*), \partial)$, more precisely,

$$(1.3) \quad C^p(K_X, F) = \bigoplus_{\mathfrak{a} \in K_X^p \setminus K_X^{p-1}} [\mathfrak{a}]^* \otimes_{\mathbb{R}} F_{b(\mathfrak{a})} \simeq (C_p(K_X, F^*))^* ,$$

and $\tilde{\partial}$ is dual to ∂ . Let $C^p(K_X/K_Y, F)$ be the maximal subset of $C^p(K_X, F)$, whose pairing with $C_p(K_Y, F^*)$ is zero. Then $(C^\bullet(K_X/K_Y, F), \tilde{\partial})$ is a sub complex of $(C^\bullet(K_X, F), \tilde{\partial})$. The following definitions are classical,

$$(1.4) \quad H^\bullet(X, F) = H^\bullet(C^\bullet(K_X, F), \tilde{\partial}) , \quad H^\bullet(X, \partial X, F) = H^\bullet(C^\bullet(K_X/K_Y, F), \tilde{\partial}) .$$

1.2. Hodge Theorem. Let g^{TX} be a Riemannian metric on X . Let h^F be a Hermitian metric on F . We identify a neighborhood of ∂X to $] -1, 0] \times Y$. Let (u, y) ($u \in] -1, 0]$, $y \in Y$) be its coordinates. We suppose that (0.1) holds.

We equip ∂X with absolute/relative boundary condition :

$$(1.5) \quad \begin{aligned} \Omega_{\text{abs}}^\bullet(X, F) &:= \left\{ \omega \in \Omega^\bullet(X, F) : i_{\frac{\partial}{\partial u}} \omega = 0 \text{ on } Y \right\} , \\ \Omega_{\text{rel}}^\bullet(X, F) &:= \left\{ \omega \in \Omega^\bullet(X, F) : du \wedge \omega = 0 \text{ on } Y \right\} . \end{aligned}$$

We write $\Omega_{\text{bd}}^\bullet(X, F)$ for short if the choice of abs/rel is clear.

Let $d^{F,*}$ be the formal adjoint of the de Rham operator d^F with respect to the L^2 -metric $\langle \cdot, \cdot \rangle_X$ (cf. §0.4). Set

$$(1.6) \quad D_X^F = d^F + d^{F,*} .$$

acting on $\Omega_{\text{bd}}^\bullet(X, F)$.

Set

$$(1.7) \quad \begin{aligned} \Omega_{\text{abs}, D^2}^\bullet(X, F) &:= \left\{ \omega \in \Omega^\bullet(X, F) : i_{\frac{\partial}{\partial u}} \omega = 0 , i_{\frac{\partial}{\partial u}} d^F \omega = 0 \text{ on } Y \right\} , \\ \Omega_{\text{rel}, D^2}^\bullet(X, F) &:= \left\{ \omega \in \Omega^\bullet(X, F) : du \wedge \omega = 0 , du \wedge d^{F,*} \omega = 0 \text{ on } Y \right\} . \end{aligned}$$

We write $\Omega_{\text{bd}, D^2}^\bullet(X, F)$ for short if the choice of abs/rel is clear.

Let $D_X^{F,2}$ act on $\Omega_{\text{bd}, D^2}^\bullet(X, F)$.

Let $\Omega_{L^2}^\bullet(X, F)$ be the completion of $\Omega^\bullet(X, F)$ with respect to $\langle \cdot, \cdot \rangle_X$.

We define the de Rham map $P_\infty : \Omega^\bullet(X, F) \rightarrow C^\bullet(K_X, F)$ by

$$(1.8) \quad P_\infty(\sigma)([\mathfrak{a}] \otimes v) = \int_{\mathfrak{a}} (\sigma, v) ,$$

for any $\mathfrak{a} \in K_X$, $v \in F_{b(\mathfrak{a})}^*$, $\sigma \in \Omega^\bullet(Z, F)$.

The following Hodge theorem is proved in [34, Proposition 4.2, Corollary 5.7] for the case $\nabla^F h^F = 0$. The fact that the same proof works in the general case is noticed in [13, Theorem 1.1].

Theorem 1.1. *We have*

$$(1.9) \quad \ker(D_X^{F,2}) = \ker(D_X^F) = \ker(d^F) \cap \ker(d^{F,*}) \cap \Omega_{\text{bd}}^\bullet(X, F) .$$

The vector space $\ker(D_X^F)$ is finite dimensional.

The following orthogonal decompositions hold,

$$(1.10) \quad \begin{aligned} \Omega_{\text{bd}}^p(X, F) &= \ker(D_X^F) \oplus d^F \Omega_{\text{bd}, D^2}^{p-1}(X, F) \oplus d^{F,*} \Omega_{\text{bd}, D^2}^{p+1}(X, F) , \\ \Omega_{L^2}^p(X, F) &= \ker(D_X^F) \oplus \overline{d^F \Omega_{\text{bd}, D^2}^{p-1}(X, F)} \oplus \overline{d^{F,*} \Omega_{\text{bd}, D^2}^{p+1}(X, F)} , \end{aligned}$$

where $\overline{}$ denotes the L^2 -closure.

For absolute (resp. relative) boundary condition, the inclusion $\ker(D_X^F) \hookrightarrow \ker(d^F) \cap \Omega_{\text{bd}}^\bullet(X, F)$ composed with the de Rham map P_∞ maps into the space of cocycles in $C^\bullet(K_X, F)$ (resp. $C^\bullet(K_X/K_Y, F)$) and we obtain an isomorphism

$$(1.11) \quad P_\infty : \ker(D_X^{F,2}) \rightarrow H_{\text{bd}}^\bullet(X, F) .$$

We define

$$(1.12) \quad H^p(\Omega_{\text{bd}}^\bullet(X, F), d^F) = \frac{\ker(d^F) \cap \Omega_{\text{bd}}^p(X, F)}{d^F \Omega_{\text{bd}}^{p-1}(X, F) \cap \Omega_{\text{bd}}^p(X, F)} .$$

Then, by Theorem 1.1, P_∞ induces the isomorphisms

$$(1.13) \quad H^p(\Omega_{\text{abs}}^\bullet(X, F), d^F) \simeq H_{\text{abs}}^p(X, F) , \quad H^p(\Omega_{\text{rel}}^\bullet(X, F), d^F) \simeq H_{\text{rel}}^p(X, F) .$$

1.3. Mayer-Vietoris sequence. Let Z be a closed \mathcal{C}^∞ -manifold. Let $i : Y \hookrightarrow Z$ be a compact hypersurface such that $Z \setminus Y = Z_1 \cup Z_2$, where Z_1, Z_2 are compact manifolds with boundary Y . Then $Z = Z_1 \cup_Y Z_2$. Let $F \rightarrow Z$ be a complex vector bundle equipped with a flat connection ∇^F . We equip ∂Z_1 (resp. ∂Z_2) with relative (resp. absolute) boundary condition. Then all the notations and results developed in the previous subsections can be applied to $(Z_1, F|_{Z_1}, \nabla^F|_{Z_1})$ and $(Z_2, F|_{Z_2}, \nabla^F|_{Z_2})$.

Let K_{Z_1}, K_{Z_2} be smooth triangulations of Z_1, Z_2 , K_Y be a smooth triangulation of Y , such that $K_Y = K_{Z_1} \cap Y = K_{Z_2} \cap Y$. Set

$$(1.14) \quad K_Z = (K_{Z_1} \setminus K_Y) \cup (K_{Z_2} \setminus K_Y) \cup K_Y .$$

Then K_Z is a smooth triangulation of Z .

The following short exact sequence holds,

$$(1.15) \quad 0 \longrightarrow (C^\bullet(K_{Z_1}/K_Y, F), \tilde{\partial}) \longrightarrow (C^\bullet(K_Z, F), \tilde{\partial}) \longrightarrow (C^\bullet(K_{Z_2}, F), \tilde{\partial}) \longrightarrow 0 .$$

It induces the long exact sequence

$$(1.16) \quad \cdots \longrightarrow H_{\text{bd}}^p(Z_1, F) \xrightarrow{\alpha_p} H^p(Z, F) \xrightarrow{\beta_p} H_{\text{bd}}^p(Z_2, F) \xrightarrow{\delta_p} \cdots .$$

If we equip Z with a Riemannian metric g^{TZ} and F with a Hermitian metric h^F . By (1.13) and (1.16), we get the long exact sequence (1.17)

$$\cdots \longrightarrow H^p(\Omega_{\text{bd}}^\bullet(Z_1, F), d^F) \xrightarrow{\alpha_p} H^p(\Omega^\bullet(Z, F), d^F) \xrightarrow{\beta_p} H^p(\Omega_{\text{bd}}^\bullet(Z_2, F), d^F) \xrightarrow{\delta_p} \cdots$$

Proposition 1.2. *The maps α_p , β_p and δ_p in (1.17) are as follows.*

- Let $[\sigma] \in H^p(\Omega_{\text{bd}}^\bullet(Z_1, F), d^F)$. There exists $\sigma' \in [\sigma]$ which vanishes on a neighborhood of Y . Extending σ' by zero, we get $\sigma'' \in \Omega^p(Z, F)$. Then $\alpha_p([\sigma]) = [\sigma'']$.
- Let $[\sigma] \in H^p(\Omega^\bullet(Z, F), d^F)$. There exists $\sigma' \in [\sigma]$ such that $\sigma'' := \sigma'|_{Z_2} \in \Omega_{\text{bd}}^\bullet(Z_2, F)$. Then $\beta_p([\sigma]) = [\sigma'']$.
- Let $[\sigma] \in H^p(\Omega_{\text{bd}}^\bullet(Z_2, F), d^F)$. There exists $\sigma' \in \Omega^\bullet(Z, F)$ such that $\sigma'|_{Z_2} \in [\sigma]$. Set $\sigma'' = d^F \sigma'|_{Z_1}$. Then $\delta_p([\sigma]) = [\sigma'']$.

Let D_Z^F be the Hodge-de Rham operator on $\Omega^\bullet(Z, F)$. Let $D_{Z_j}^F$ ($j = 1, 2$) be the Hodge-de Rham operator on $\Omega_{\text{bd}}^\bullet(Z_j, F)$. Set

$$(1.18) \quad \mathcal{H}^\bullet(Z, F) = \ker D_Z^F, \quad \mathcal{H}_{\text{bd}}^\bullet(Z_j, F) = \ker D_{Z_j}^F, \quad \text{for } j = 1, 2.$$

Applying Theorem 1.1, (1.16) induces the following long exact sequence

$$(1.19) \quad \longrightarrow \mathcal{H}_{\text{bd}}^p(Z_1, F) \xrightarrow{\alpha_p} \mathcal{H}^p(Z, F) \xrightarrow{\beta_p} \mathcal{H}_{\text{bd}}^p(Z_2, F) \xrightarrow{\delta_p} ,$$

which is isomorphic to (1.17).

We recall that $\langle \cdot, \cdot \rangle$ is defined in §0.4.

The following proposition is a consequence of Theorem 1.1 and Proposition 1.2.

Proposition 1.3. *For $\omega \in \mathcal{H}_{\text{bd}}^p(Z_1, F)$ and $\mu \in \mathcal{H}^p(Z, F)$, we have*

$$(1.20) \quad \langle \alpha_p(\omega), \mu \rangle_Z = \langle \omega, \mu \rangle_{Z_1}.$$

For $\omega \in \mathcal{H}^p(Z, F)$ and $\mu \in \mathcal{H}_{\text{bd}}^p(Z_2, F)$, we have

$$(1.21) \quad \langle \beta_p(\omega), \mu \rangle_{Z_2} = \langle \omega, \mu \rangle_{Z_2}.$$

For $\omega \in \mathcal{H}_{\text{bd}}^p(Z_2, F)$ and $\mu \in \mathcal{H}_{\text{bd}}^{p+1}(Z_{1,R}, F)$, we have

$$(1.22) \quad \langle \delta_p(\omega), \mu \rangle_{Z_1} = \langle \omega, i_{\frac{\partial}{\partial u}} \mu \rangle_Y.$$

2. HODGE-DE RHAM OPERATOR ON MANIFOLD WITH CYLINDRICAL ENDS

Let Z_∞ be a Riemannian manifold with cylindrical ends, more precisely, there is an isometric inclusion $\mathbb{R}_+ \times Y \subseteq Z_\infty$ with Y closed and $Z_\infty \setminus (\mathbb{R}_+ \times Y)$ compact. In this section, we review some spectral properties about the Hodge Laplacian on Z_∞ .

In §2.1, we consider the Hodge-de Rham operator acting on a closed manifold together with an additional odd Grassmannian variable du . In later subsections, u will serve as the coordinate on \mathbb{R}_+ . In §2.2, we study the eigensections of the Hodge-de Rham operator acting on $I \times Y$ with I a bounded open interval. In §2.3, we concentrate on studying the generalized eigensections of the Hodge-de Rham operator acting on Z_∞ . In particular, (following [31]) we define the scattering matrix and state its link to generalized eigensections. In §2.4, we study the generalized eigensections associated to eigenvalue 0 (called extended L^2 -solutions).

2.1. Hodge-de Rham operator with an additional odd Grassmannian variable. Let Y be a closed \mathcal{C}^∞ -manifold. Let (F, ∇^F) be a flat complex vector bundle over Y . Let g^{TY} be a Riemannian metric on Y . Let h^F be a Hermitian metric on F . Let D_Y^F be the Hodge-de Rham operator (defined in §0.4) acting on $\Omega^\bullet(Y, F)$.

Set

$$(2.1) \quad \mathcal{H}^\bullet(Y, F) = \ker D_Y^{F,2}.$$

For any $\mu \in \mathbb{R}$, let $\mathcal{E}_\mu(Y, F)$ be the eigenspace of D_Y^F associated to eigenvalue μ .

Let du be an additional odd Grassmannian variable, such that $(du)^2 = 0$. Let $\Omega^\bullet(Y, F[du])$ be the algebra generated by $\Omega^\bullet(Y, F)$ and du , i.e.,

$$(2.2) \quad \Omega^\bullet(Y, F[du]) = \Omega^\bullet(Y, F) \oplus \Omega^\bullet(Y, F)du.$$

We equip $\Omega^\bullet(Y, F[du])$ with a grading : the degree p component is $\Omega^p(Y, F) \oplus \Omega^{p-1}(Y, F)du$.

The L^2 -norm $\|\cdot\|_Y$ and its associated scalar product $\langle \cdot, \cdot \rangle_Y$ on $\Omega^\bullet(Y, F)$ (defined in §0.4) extend to $\Omega^\bullet(Y, F[du])$, such that, for any $\tau_0, \tau_1 \in \Omega^\bullet(Y, F)$,

$$(2.3) \quad \|\tau_0 + du \wedge \tau_1\|_Y^2 = \|\tau_0\|_Y^2 + \|\tau_1\|_Y^2.$$

We define actions $du \wedge$, $i_{\frac{\partial}{\partial u}}$ and $c(\frac{\partial}{\partial u})$ on $\Omega^\bullet(Y, F[du])$, such that, for any $\tau_0, \tau_1 \in \Omega^\bullet(Y, F)$,

$$(2.4) \quad du \wedge (\tau_0 + du \wedge \tau_1) = du \wedge \tau_0, \quad i_{\frac{\partial}{\partial u}}(\tau_0 + du \wedge \tau_1) = \tau_1, \quad c(\frac{\partial}{\partial u}) = du \wedge -i_{\frac{\partial}{\partial u}}.$$

The action of D_Y^F extends to $\Omega^\bullet(Y, F[du])$ by anti-commuting with du , i.e.,

$$(2.5) \quad D_Y^F(du \wedge \tau) = -du \wedge D_Y^F \tau, \quad \text{for } \tau \in \Omega^\bullet(Y, F).$$

Let $\mathcal{H}^\bullet(Y, F[du])$ be the kernel of this extended action. Let $\mathcal{E}_\mu(Y, F[du])$ be the eigenspace of the extended action associated to the eigenvalue μ , then

$$(2.6) \quad \begin{aligned} \mathcal{H}^\bullet(Y, F[du]) &= \mathcal{H}^\bullet(Y, F) \oplus \mathcal{H}^\bullet(Y, F)du, \\ \mathcal{E}_\mu(Y, F[du]) &= \mathcal{E}_\mu(Y, F) \oplus \mathcal{E}_{-\mu}(Y, F)du. \end{aligned}$$

We remark that the action of $c(\frac{\partial}{\partial u})$ exchanges $\mathcal{E}_{\pm\mu}(Y, F[du])$.

2.2. Hodge-de Rham operator on a cylinder. Let $I =]a, b[\subseteq \mathbb{R}$ be an interval. We consider the cylinder $I \times Y$ with coordinates (u, y) ($u \in I, y \in Y$). Let $\pi_Y : I \times Y \rightarrow Y$ be the natural projection. We equip $I \times Y$ with the product metric (cf. (0.1)).

The pull back of F by π_Y is a flat vector bundle on $I \times Y$, which is still denoted F . Its flat connection is defined by

$$(2.7) \quad \nabla^F = du \wedge \frac{\partial}{\partial u} + \nabla^F|_Y.$$

And the pull back metric on F is still denoted h^F .

We have the canonical identification

$$(2.8) \quad \Omega^\bullet(I \times Y, F) \simeq \mathcal{C}^\infty(I, \Omega^\bullet(Y, F[du])).$$

For any $\omega \in \Omega^\bullet(I \times Y, F)$, $u \in I$, let $\omega_u \in \Omega^\bullet(Y, F[du])$ be the value of the corresponding function at u . For any $\tau \in \Omega^\bullet(Y, F[du])$, let $\pi_Y^* \tau \in \Omega^\bullet(I \times Y, F)$ be the differential form associated to the constant function of value τ . Then, for any $\omega, \omega' \in \Omega^\bullet(I \times Y, F)$,

$$(2.9) \quad \langle \omega, \omega' \rangle_{I \times Y} = \int_I \langle \omega_u, \omega'_u \rangle_Y du.$$

Let D_{IY}^F be the Hodge-de Rham operator acting on $\Omega^\bullet(I \times Y, F)$, then,

$$(2.10) \quad D_{IY}^F = c\left(\frac{\partial}{\partial u}\right) \frac{\partial}{\partial u} + D_Y^F.$$

By the Green Formula, for any $\omega_1, \omega_2 \in \Omega^\bullet(I \times Y, F)$, we have

$$(2.11) \quad \langle D_{IY}^F \omega_1, \omega_2 \rangle_{I \times Y} - \langle \omega_1, D_{IY}^F \omega_2 \rangle_{I \times Y} = \langle c\left(\frac{\partial}{\partial u}\right) \omega_{1,b}, \omega_{2,b} \rangle_Y - \langle c\left(\frac{\partial}{\partial u}\right) \omega_{1,a}, \omega_{2,a} \rangle_Y.$$

Set

$$(2.12) \quad \delta_Y = \min\{|\mu| : \mu \in \text{Sp}(D_Y^F) \setminus \{0\}\}.$$

Let $\omega \in \Omega^\bullet(I \times Y, F)$ such that $D_{IY}^F \omega = \lambda \omega$ with $|\lambda| < \delta_Y$. Then,

$$(2.13) \quad \begin{aligned} \omega &= e^{-iu\lambda} (\phi_0^- - ic\left(\frac{\partial}{\partial u}\right) \phi_0^-) + e^{iu\lambda} (\phi_0^+ + ic\left(\frac{\partial}{\partial u}\right) \phi_0^+) \\ &+ \sum_{\mu \neq 0} \left\{ e^{-\sqrt{\mu^2 - \lambda^2}u} \left(\phi_\mu^- - \frac{\mu - \lambda}{\sqrt{\mu^2 - \lambda^2}} c\left(\frac{\partial}{\partial u}\right) \phi_\mu^- \right) + e^{\sqrt{\mu^2 - \lambda^2}u} \left(\phi_\mu^+ + \frac{\mu - \lambda}{\sqrt{\mu^2 - \lambda^2}} c\left(\frac{\partial}{\partial u}\right) \phi_\mu^+ \right) \right\}, \end{aligned}$$

where $\mu \in \text{Sp}(D_Y^F)$, $\phi_0^\pm \in \mathcal{H}^\bullet(Y, F)$, $\phi_\mu^\pm \in \mathcal{E}_\mu^\bullet(Y, F[du])$ (as convention, $\phi_\mu^\pm = 0$ for $\mu \notin \text{Sp}(D_Y^F)$). Set

$$(2.14) \quad \omega^{\text{zm}, \pm} = e^{\pm iu\lambda} (\phi_0^\pm \pm ic\left(\frac{\partial}{\partial u}\right) \phi_0^\pm), \quad \omega^{\text{zm}} = \omega^{\text{zm}, -} + \omega^{\text{zm}, +}.$$

The ω^{zm} is called the zeromode of ω . Set

$$(2.15) \quad \begin{aligned} \omega^{\mu, \pm} &= e^{\pm \sqrt{\mu^2 - \lambda^2}u} \left(\phi_\mu^\pm \pm \frac{\mu - \lambda}{\sqrt{\mu^2 - \lambda^2}} c\left(\frac{\partial}{\partial u}\right) \phi_\mu^\pm \right), \quad \omega^\mu = \omega^{\mu, -} + \omega^{\mu, +}, \\ \omega^\pm &= \sum_{\mu \neq 0} \omega^{\mu, \pm}, \quad \omega^{\text{nz}} = \omega^- + \omega^+. \end{aligned}$$

We have the following decomposition

$$(2.16) \quad \omega = \omega^{\text{zm}} + \omega^{\text{nz}} = \omega^{\text{zm}} + \sum_{\mu \neq 0} (\omega^{\mu, +} + \omega^{\mu, -}).$$

Furthermore, the above decomposition is fiberwise orthogonal, i.e., for any $u \in I$, and $\mu' \neq \mu$, we have

$$(2.17) \quad \langle \omega_u^{\text{zm}}, \omega_u^{\mu, +} + \omega_u^{\mu, -} \rangle_Y = 0, \quad \langle \omega_u^{\mu, +} + \omega_u^{\mu, -}, \omega_u^{\mu', +} + \omega_u^{\mu', -} \rangle_Y = 0.$$

For any $a < u < v < b$, a simple estimation yields

$$(2.18) \quad \|\omega_v^-\|_Y \leq e^{-(v-u)\sqrt{\delta_Y^2 - \lambda^2}} \|\omega_u^-\|_Y, \quad \|\omega_u^+\|_Y \leq e^{-(v-u)\sqrt{\delta_Y^2 - \lambda^2}} \|\omega_v^+\|_Y.$$

By (2.4) and (2.14), $\|\omega_u^{\text{zm}}\|_Y$ does not depend on $u \in I$. We denote

$$(2.19) \quad \|\omega^{\text{zm}}\|_Y = \|\omega_u^{\text{zm}}\|_Y.$$

Lemma 2.1. For any eigensections $\omega_1, \omega_2 \in \Omega^\bullet(I \times Y, F)$ with eigenvalue $\lambda \in]-\delta_Y, \delta_Y[$, we have

$$(2.20) \quad \begin{aligned} \langle \omega_1^{\text{nz}}, \omega_2^{\text{nz}} \rangle_{I \times Y} &\leq \left(\frac{1}{1 - e^{-\sqrt{\delta_Y^2 - \lambda^2}(b-a)}} \right)^2 \cdot \frac{1}{\sqrt{\delta_Y^2 - \lambda^2}} \cdot \|\omega_1\|_{\partial(I \times Y)} \cdot \|\omega_2\|_{\partial(I \times Y)}, \\ \langle \omega_1^{\text{zm}}, \omega_2^{\text{zm}} \rangle_Y &\leq \frac{1}{2} \|\omega_1\|_{\partial(I \times Y)} \cdot \|\omega_2\|_{\partial(I \times Y)}. \end{aligned}$$

Proof. The first inequality in (2.20) comes from (2.9), (2.12), (2.15), (2.17) and Cauchy's inequality. The second inequality in (2.20) comes from (2.19). \square

2.3. Spectrum of Hodge-de Rham operator on manifold with cylindrical ends. Still, let (Y, g^{TY}) be a closed Riemannian manifold. Let $(Z_\infty, g^{TZ_\infty})$ be a non-compact complet manifold with cylindrical end Y , i.e., there exists a subset $U \subseteq Z_\infty$, which is isometric to $\mathbb{R}_+ \times Y$, and $Z_\infty \setminus U$ is compact.

Let (F, ∇^F) be a flat complex vector bundle over Z_∞ . Using parallel transport along $\frac{\partial}{\partial u}$, $(F|_U, \nabla^F|_U)$ is identified to $\pi_Y^*(F|_Y, \nabla^F|_Y)$, i.e., (2.7) holds. Let h^F be a Hermitian metric on F . We suppose that $(F|_U, h^F|_U)$ satisfies (0.1).

Let $D_{Z_\infty}^F$ be the Hodge-de Rham operator acting on $\Omega_c^\bullet(Z_\infty, F)$. By [31, Theorem 3.2], $D_{Z_\infty}^F$ is essentially self-adjoint. Its self-adjoint extension is still denoted by $D_{Z_\infty}^F$. Let $\Omega_{L^2}^\bullet(Z_\infty, F)$ be L^2 -completion of $\Omega_c^\bullet(Z_\infty, F)$, then

$$(2.21) \quad \Omega_{L^2}^\bullet(Z_\infty, F) = \mathcal{E}_{\text{pp}}^\bullet(Z_\infty, F) \oplus \mathcal{E}_{\text{sc}}^\bullet(Z_\infty, F) \oplus \mathcal{E}_{\text{ac}}^\bullet(Z_\infty, F),$$

where the vector spaces on the right hand side are, sequentially, associated with purely point (p.p.) spectrum, singularly continuous (s.c.) spectrum and absolutely continuous (a.c.) spectrum of $D_{Z_\infty}^F$ (cf. [35, chapter 7.2]). Let $D_{Z_\infty, \text{pp}}^F$, $D_{Z_\infty, \text{sc}}^F$ and $D_{Z_\infty, \text{ac}}^F$ be the corresponding restriction of $D_{Z_\infty}^F$.

For $\lambda \in \mathbb{R}$, let $\mathcal{E}_\lambda \subseteq \Omega^\bullet(Z_\infty, F)$ be the vector space of generalized eigensections of $D_{Z_\infty}^F$ with eigenvalue λ (cf. [3, Chapter 5]). For the moment, it is sufficient to understand $(\mathcal{E}_\lambda)_{\lambda \in \mathbb{R}}$ as a family of subspaces of $\Omega^\bullet(Z_\infty, F)$ satisfying :

- for any $\omega_\lambda \in \mathcal{E}_\lambda$, we have $D_{Z_\infty}^F \omega_\lambda = \lambda \omega_\lambda$;
- for any $\omega \in \mathcal{E}_{\text{ac}}^\bullet(Z_\infty, F) \cap \Omega^\bullet(Z_\infty, F)$, there exists a smooth family $\omega_\lambda \in \mathcal{E}_\lambda$, such that $\omega = \int_{\mathbb{R}} \omega_\lambda d\lambda$.

By definition, we have $\mathcal{E}_\lambda \cap \Omega_{L^2}^\bullet(Z_\infty, F) = 0$. As a consequence, a generalized eigensection is determined by its restriction to the cylinder part.

On the cylinder part of Z_∞ , all the analysis done in §2.2 are still valid. And we will continue to use the terminologies zeromode, non-zeromode, etc.

Before describing these spaces in more detail, we need a model operator. We recall that $\Omega^\bullet(Y, F[du])$, $\mathcal{H}^\bullet(Y, F)$ and $\mathcal{E}_\mu(Y, F)$ are defined in §2.1. Let

$$(2.22) \quad \Pi : \Omega^\bullet(Y, F[du]) \rightarrow \mathcal{H}^\bullet(Y, F)du \oplus \bigoplus_{\mu > 0} \left((1 - du)\mathcal{E}_\mu(Y, F) \oplus (1 + du)\mathcal{E}_{-\mu}(Y, F) \right)$$

be the orthogonal projection. Set

$$(2.23) \quad \Omega_\Pi^\bullet(\mathbb{R}_+ \times Y, F) = \left\{ \omega \in \Omega^\bullet(\mathbb{R}_+ \times Y, F) : \omega_0 \in \ker(\Pi) \right\},$$

where $\omega_0 = \omega_u|_{u=0} \in \Omega^\bullet(Y, F[du])$ is defined in §2.2. Let $D_{\mathbb{R}_+ \times Y}^F$ be the Hodge-de Rham operator on $\mathbb{R}_+ \times Y$ with domain $\Omega_\Pi^\bullet(\mathbb{R}_+ \times Y, F)$. Then, $D_{\mathbb{R}_+ \times Y}^F$ has only a.c. spectrum.

Remark 2.2. The above boundary condition is introduced by Atiyah-Patodi-Singer [1].

Let $j : \mathbb{R}_+ \times Y \hookrightarrow Z_\infty$ be the canonical inclusion. Then j induces the inclusion

$$(2.24) \quad J : \Omega_{L^2}^\bullet(\mathbb{R}_+ \times Y, F) \hookrightarrow \Omega_{L^2}^\bullet(Z_\infty, F).$$

We define the wave operators

$$(2.25) \quad W_{\pm} (D_{Z_{\infty}}^F, D_{\mathbb{R}_+ Y}^F) = \lim_{t \rightarrow \pm \infty} e^{itD_{Z_{\infty}}^F} J e^{-itD_{\mathbb{R}_+ Y}^F} .$$

By [31, Proposition 4.9], $W_{\pm} (D_{Z_{\infty}}^F, D_{\mathbb{R}_+ Y}^F)$ are well-defined.

Müller [31, Theorem 4.1, Theorem 4.10] established the following theorem.

Theorem 2.3. *The operator $D_{Z_{\infty}}^F$ has no singularly continuous spectrum.*

For any $t > 0$, the operator $\exp \left(-tD_{Z_{\infty}, \text{pp}}^{F,2} \right)$ is of trace class.

The wave operator $W_{\pm} (D_{Z_{\infty}}^F, D_{\mathbb{R}_+ Y}^F)$ gives a unitary equivalence between $D_{\mathbb{R}_+ Y}^F$ and $D_{Z_{\infty}, \text{ac}}^F$, i.e., $W_{\pm} (D_{Z_{\infty}}^F, D_{\mathbb{R}_+ Y}^F)$ is a unitary map from $\Omega_{L^2}^{\bullet}(\mathbb{R}_+ \times Y, F)$ to $\mathcal{E}_{\text{ac}}^{\bullet}(Z_{\infty}, F)$, and the following diagram commutes.

$$(2.26) \quad \begin{array}{ccc} \Omega_{L^2}^{\bullet}(\mathbb{R}_+ \times Y, F) & \xrightarrow{D_{\mathbb{R}_+ Y}^F} & \Omega_{L^2}^{\bullet}(\mathbb{R}_+ \times Y, F) \\ W_{\pm}(D_{Z_{\infty}}^F, D_{\mathbb{R}_+ Y}^F) \downarrow & & \downarrow W_{\pm}(D_{Z_{\infty}}^F, D_{\mathbb{R}_+ Y}^F) \\ \mathcal{E}_{\text{ac}}^{\bullet}(Z_{\infty}, F) & \xrightarrow{D_{Z_{\infty}}^F} & \mathcal{E}_{\text{ac}}^{\bullet}(Z_{\infty}, F) \end{array}$$

Set

$$(2.27) \quad C(D_{Z_{\infty}}^F, D_{\mathbb{R}_+ Y}^F) = W_+^* (D_{Z_{\infty}}^F, D_{\mathbb{R}_+ Y}^F) W_- (D_{Z_{\infty}}^F, D_{\mathbb{R}_+ Y}^F) ,$$

which acts on $\Omega_{L^2}^{\bullet}(\mathbb{R}_+ \times Y, F)$. Then, $C(D_{Z_{\infty}}^F, D_{\mathbb{R}_+ Y}^F)$ commutes with $D_{\mathbb{R}_+ Y}^F$.

We remark that any generalized eigensection of $D_{\mathbb{R}_+ Y}^F$ with eigenvalue $\lambda \in] -\delta_Y, \delta_Y[$ takes the form

$$(2.28) \quad E_0(\phi, \lambda) = e^{-i\lambda u}(\phi - ic(\frac{\partial}{\partial u})\phi) + e^{i\lambda u}(\phi + ic(\frac{\partial}{\partial u})\phi) ,$$

with $\phi \in \mathcal{H}^{\bullet}(Y, F)$.

Definition 2.4. Since $C(D_{Z_{\infty}}^F, D_{\mathbb{R}_+ Y}^F)$ preserves the spectral decomposition of $\Omega_{L^2}^{\bullet}(\mathbb{R}_+ \times Y, F)$ with respect to $D_{\mathbb{R}_+ Y}^F$, for any $\lambda \in] -\delta_Y, \delta_Y[$, there exists uniquely $C(\lambda) \in \text{End}(\mathcal{H}^{\bullet}(Y, F))$, such that

$$(2.29) \quad C(D_{Z_{\infty}}^F, D_{\mathbb{R}_+ Y}^F) E_0(\phi, \lambda) = E_0(C(\lambda)\phi, \lambda) .$$

We extend the action of $C(\lambda)$ to $\mathcal{H}^{\bullet}(Y, F[du])$ by demanding

$$(2.30) \quad C(\lambda)c(\frac{\partial}{\partial u}) = -c(\frac{\partial}{\partial u})C(\lambda) .$$

The $(C(\lambda))_{\lambda}$ is called the scattering matrix associated to $D_{Z_{\infty}}^F$.

The following property is stated in [31, §4].

Proposition 2.5. *Each generalized eigensection of $D_{Z_{\infty}, \text{ac}}^F$ with eigenvalue $\lambda \in] -\delta_Y, \delta_Y[$ takes the following form over $\mathbb{R}_+ \times Y \simeq U \subseteq Z_{\infty}$:*

$$(2.31) \quad E(\phi, \lambda) = e^{-i\lambda u}(\phi - ic(\frac{\partial}{\partial u})\phi) + e^{i\lambda u}C(\lambda)(\phi - ic(\frac{\partial}{\partial u})\phi) + \theta(\phi, \lambda) ,$$

where $\phi \in \mathcal{H}^{\bullet}(Y, F)$ and $\theta(\phi, \lambda) \in \Omega_{L^2}^{\bullet}(\mathbb{R}_+ \times Y, F)$, and, for any $u \in \mathbb{R}_+$,

$$(2.32) \quad \theta_u(\phi, \lambda) \perp \mathcal{H}^{\bullet}(Y, F[du]) .$$

Conversely, for any $\phi \in \mathcal{H}^{\bullet}(Y, F)$ and $\lambda \in] -\delta_Y, \delta_Y[$, there exists a unique generalized eigensection $E(\phi, \lambda)$ of $D_{Z_{\infty}, \text{ac}}^F$ satisfying (2.31).

We remark that $E(\phi, \lambda)$ depends linearly on ϕ and analytically on λ (cf. [31, §4]). Since $\mathcal{H}^\bullet(Y, F)$ is finite dimensional, by an argument of compactness, there exists $C > 0$, such that, for any $\phi \in \mathcal{H}^\bullet(Y, F)$ and $\lambda \in]-\delta_Y/2, \delta_Y/2[$, we have

$$(2.33) \quad \|E(\phi, \lambda)\|_{Z_\infty \setminus U} \leq C \|\phi\|_Y.$$

We list below several properties of $C(\lambda)$ (cf. [31, §4]).

Proposition 2.6. *The following properties hold*

- $C(\lambda)$ depends analytically on λ ;
- $C(\lambda) \in \text{End}(\mathcal{H}^\bullet(Y, F[du]))$ is unitary ;
- $C(\lambda)$ preserves $\mathcal{H}^p(Y, F)$ and $\mathcal{H}^p(Y, F)du$ for any p ;
- $C(\lambda)C(-\lambda) = 1$, in particular, $C(0)^2 = 1$.

2.4. Extended L^2 -solutions. Set

$$(2.34) \quad \mathcal{H}_{L^2}^\bullet(Z_\infty, F) = \Omega_{L^2}^\bullet(Z_\infty, F) \cap \ker(D_{Z_\infty}^{F,2}),$$

The elements of $\mathcal{H}_{L^2}^\bullet(Z_\infty, F)$ are called L^2 -solutions of $D_{Z_\infty}^{F,2}\omega = 0$.

We recall that the decomposition $\omega = \omega^{\text{zm}} + \omega^{\text{zn}} = \omega^{\text{zm}} + \omega^- + \omega^+$ is defined in (2.16).

Definition 2.7. Set

$$(2.35) \quad \mathcal{H}^\bullet(Z_\infty, F) = \{(\omega, \hat{\omega}) \in \ker(D_{Z_\infty}^{F,2}) \oplus \mathcal{H}^\bullet(Y, F[du]) : \omega^+ = 0, \omega^{\text{zm}} = \pi_Y^* \hat{\omega}\},$$

The elements of $\mathcal{H}^\bullet(Z_\infty, F)$ are called extended L^2 -solutions of $D_{Z_\infty}^{F,2}\omega = 0$.

Remark 2.8. In fact, $\mathcal{H}^\bullet(Z_\infty, F)$ is the vector space spanned by $\mathcal{H}_{L^2}^\bullet(Z_\infty, F)$ and generalized eigensections of $D_{Z_\infty}^F$ associated with eigenvalue $\lambda = 0$, i.e.,

$$(2.36) \quad \mathcal{H}^\bullet(Z_\infty, F) = \mathcal{H}_{L^2}^\bullet(Z_\infty, F) \oplus \{E(\phi, 0) : \phi \in \mathcal{H}^\bullet(Y, F)\},$$

where $E(\phi, 0) = E(\phi, \lambda)|_{\lambda=0}$ is given by (2.31).

Proposition 2.9. *For any $(\omega, \hat{\omega}) \in \mathcal{H}^\bullet(Z_\infty, F)$, we have*

$$(2.37) \quad d^F \omega = d^{F,*} \omega = 0.$$

Proof. By (2.13), both $d^F \omega$ and $d^{F,*} \omega$ are L^2 -sections, which are orthogonal with respect to the L^2 -metric. Then $d^F \omega + d^{F,*} \omega = D^F \omega = 0$ implies (2.37). \square

Comparing (2.13) and Proposition 2.9, we get the following decomposition of $(\omega, \hat{\omega}) \in \mathcal{H}^\bullet(Z_\infty, F)$ on the cylinder U ,

$$(2.38) \quad \omega|_U = \pi_Y^* \hat{\omega} + \sum_{\mu > 0, \mu \in \text{Sp}(D_Y^F)} e^{-\mu u} (\tau_{\mu,1} - du \wedge \tau_{\mu,2}),$$

where $\tau_{\mu,1} \in \Omega^\bullet(Y, F)$, $\tau_{\mu,2} \in \Omega^{\bullet-1}(Y, F)$, and

$$(2.39) \quad d^F \tau_{\mu,1} = d^{F,*} \tau_{\mu,2} = 0, \quad d^{F,*} \tau_{\mu,1} = \mu \tau_{\mu,2}, \quad d^F \tau_{\mu,2} = \mu \tau_{\mu,1}.$$

Definition 2.10. We define

$$(2.40) \quad \begin{aligned} \mathcal{R}_{d^F} : \mathcal{H}^\bullet(Z_\infty, F) &\rightarrow \Omega^{\bullet-1}(\mathbb{R}_+ \times Y, F), \\ \mathcal{R}_{d^{F,*}} : \mathcal{H}^\bullet(Z_\infty, F) &\rightarrow \Omega^{\bullet+1}(\mathbb{R}_+ \times Y, F), \end{aligned}$$

such that, for any $(\omega, \hat{\omega}) \in \mathcal{H}^\bullet(Z_\infty, F)$, whose expansion is given by (2.38), we have

$$(2.41) \quad \mathcal{R}_{d^F}(\omega, \hat{\omega}) = \sum_{\mu>0} \frac{1}{\mu} e^{-\mu u} \tau_{\mu,2} , \quad \mathcal{R}_{d^{F,*}}(\omega, \hat{\omega}) = \sum_{\mu>0} \frac{1}{\mu} e^{-\mu u} du \wedge \tau_{\mu,1} .$$

Proposition 2.11. *The following identities hold :*

$$(2.42) \quad \begin{aligned} d^F \mathcal{R}_{d^F}(\omega, \hat{\omega}) &= \omega|_{\mathbb{R}_+ \times Y} - \pi_Y^* \hat{\omega} , & d^{F,*} \mathcal{R}_{d^F}(\omega, \hat{\omega}) &= 0 , \\ d^{F,*} \mathcal{R}_{d^{F,*}}(\omega, \hat{\omega}) &= \omega|_{\mathbb{R}_+ \times Y} - \pi_Y^* \hat{\omega} , & d^F \mathcal{R}_{d^{F,*}}(\omega, \hat{\omega}) &= 0 . \end{aligned}$$

Proof. These are direct consequences of (2.38), (2.39) and (2.41). \square

Definition 2.12. Set

$$(2.43) \quad \mathcal{L}^\bullet = \left\{ \hat{\omega} \in \mathcal{H}^\bullet(Y, F[du]) : \text{there exists } \omega \text{ such that } (\omega, \hat{\omega}) \in \mathcal{H}^\bullet(Z_\infty, F) \right\} ,$$

called the set of limiting values of $\mathcal{H}^\bullet(Z_\infty, F)$.

Still, $C(\lambda)$ is the scattering matrix. We denote $C = C(0)$. By (2.31), Proposition 2.6 and the fact that $\mathcal{L}^\bullet = \bigoplus \mathcal{L}^p$, we see that

$$(2.44) \quad \mathcal{L}^\bullet = \text{Im}(C + 1) = \ker(C - 1) .$$

Furthermore, let $P_{\mathcal{L}} : \mathcal{H}^\bullet(Y, F[du]) \rightarrow \mathcal{L}^\bullet$ be the orthogonal projection, then

$$(2.45) \quad C = 2P_{\mathcal{L}} - 1 .$$

We recall that the operator $i \frac{\partial}{\partial u}$ acting on $\mathcal{H}^\bullet(Y, F[du])$ is defined by (2.4). As consequences of (2.30), (2.44) and Proposition 2.6, there exist $\mathcal{L}_{\text{abs}}^p \subseteq \mathcal{H}^p(Y, F)$ and $\mathcal{L}_{\text{rel}}^p \subseteq \mathcal{H}^{p-1}(Y, F)du$ such that

$$(2.46) \quad \mathcal{L}^p = \mathcal{L}_{\text{abs}}^p \oplus \mathcal{L}_{\text{rel}}^p , \quad \mathcal{L}_{\text{abs}}^{p,\perp} = i \frac{\partial}{\partial u} \mathcal{L}_{\text{rel}}^{p+1} ,$$

where $\mathcal{L}_{\text{abs}}^{p,\perp} \subseteq \mathcal{H}^p(Y, F)$ is the orthogonal complement of $\mathcal{L}_{\text{abs}}^p$. We call $\mathcal{L}_{\text{abs/rel}}^\bullet$ the absolute/relative component of \mathcal{L}^\bullet .

We have the obvious short exact sequence

$$(2.47) \quad 0 \longrightarrow \mathcal{H}_{L^2}^\bullet(Z_\infty, F) \longrightarrow \mathcal{H}^\bullet(Z_\infty, F) \longrightarrow \mathcal{L}^\bullet \longrightarrow 0 .$$

We denote

$$(2.48) \quad \mathcal{H}_{\text{abs/rel}}^\bullet(Z_\infty, F) = \left\{ (\omega, \hat{\omega}) \in \mathcal{H}^\bullet(Z_\infty, F) : \hat{\omega} \in \mathcal{L}_{\text{abs/rel}}^\bullet \right\} .$$

Then we have the following short exact sequence

$$(2.49) \quad 0 \longrightarrow \mathcal{H}_{L^2}^\bullet(Z_\infty, F) \longrightarrow \mathcal{H}_{\text{abs/rel}}^\bullet(Z_\infty, F) \longrightarrow \mathcal{L}_{\text{abs/rel}}^\bullet \longrightarrow 0 .$$

3. ASYMPTOTIC PROPERTIES OF THE SPECTRUM

We recall that Z_R , F and $D_{Z_R}^F$ are defined in §0.2. In this section, we study the asymptotic behavior of $\text{Sp}(D_{Z_R}^F)$ as $R \rightarrow \infty$.

In §3.1, we construct Z_R . In §3.2, we construct a model space of the eigensections of $D_{Z_R}^F$. In §3.3, we estimate the kernel of $D_{Z_R}^{F,2}$. In §3.4, we estimate the small eigenvalues of $D_{Z_R}^F$.

3.1. Gluing of two manifolds with the same boundary. Let Z be a closed manifold. Let $i : Y \hookrightarrow Z$ be a compact hypersurface such that $Z \setminus Y = Z_1 \cup Z_2$, where Z_1, Z_2 are compact manifolds with boundary Y . Then $Z = Z_1 \cup_Y Z_2$.

Let $U_j \subseteq Z_j$ ($j = 1, 2$) be a collar neighborhood of $\partial Z_j \simeq Y$, more precisely, we fix the diffeomorphisms

$$(3.1) \quad i_1 :]-1, 0] \times Y \rightarrow U_1, \quad i_2 : [0, 1[\times Y \rightarrow U_2,$$

such that $i_j(\{0\} \times Y) = \partial Z_j$ ($j = 1, 2$). Set $U = U_1 \cup_Y U_2 \subseteq Z$, then, i_1 and i_2 induce the identification

$$(3.2) \quad i :]-1, 1[\times Y \rightarrow U \subseteq Z.$$

Let (F, ∇^F) be a flat vector bundle on Z .

Let g^{TZ} be a Riemannian metric on Z . Let h^F be a Hermitian metric on F . We suppose that (0.1) holds.

Set

$$(3.3) \quad \begin{aligned} Z_{1,R} &= Z_1 \cup_Y [0, R] \times Y, & Z_{2,R} &= Z_2 \cup_Y [-R, 0] \times Y, & \text{for } 0 \leq R \leq \infty, \\ Z_{1,\infty} &= Z_1 \cup_Y [0, \infty[\times Y, & Z_{2,\infty} &= Z_2 \cup_Y]-\infty, 0] \times Y. \end{aligned}$$

where the gluing identifies $\partial Z_j \simeq Y$ ($j = 1, 2$) to $\{0\} \times Y$. For any $0 \leq R < \infty$, we define

$$(3.4) \quad \begin{aligned} f_R : [0, 2R] \times Y &\rightarrow [-2R, 0] \times Y \\ (u, y) &\mapsto (u - 2R, y). \end{aligned}$$

Set

$$(3.5) \quad Z_R = Z_{1,2R} \cup_{f_R} Z_{2,2R} = Z_{1,R} \cup_Y Z_{2,R}.$$

Then (F, ∇^F) extends to a flat vector bundle on Z_R by respecting (2.7), g^{TZ} and h^F extend to Z_R by respecting (0.1).

In the sequel, all the canonical projections from $[-R, 0] \times Y$, $[0, R] \times Y$ and $[-R, R] \times Y$ ($0 \leq R \leq \infty$) onto Y will simply be denoted π_Y if there is no confusion.

In the sequel, for any $0 \leq R \leq \infty$, $[0, R] \times Y \subseteq Z_{1,R}$ (resp. $[-R, 0] \times Y \subseteq Z_{2,R}$), the cylindrical part of $Z_{1,R}$ (resp. $Z_{2,R}$), will be referred as $I_{1,R}Y$ (resp. $I_{2,R}Y$); if $R < \infty$, the cylindrical part of Z_R , i.e., the gluing of $I_{1,R}Y$ and $I_{2,R}Y$, will be referred as I_RY . On $I_{1,R}Y$, we use coordinates (u_1, y) with $u_1 \in [0, R]$, $y \in Y$; on $I_{2,R}Y$, we use coordinates (u_2, y) with $u_2 \in [-R, 0]$, $y \in Y$; on I_RY , we use coordinates (u, y) with $u \in [-R, R]$, $y \in Y$. Under the identifications $I_{1,2R}Y \simeq I_{2,2R}Y \simeq I_RY$ induced by (3.5), and the transformation of coordinates is given by

$$(3.6) \quad u = u_1 - R = u_2 + R.$$

For $A \subseteq \mathbb{R}$, set

$$(3.7) \quad \begin{aligned} I_{j,R}Y(A) &= \{(u_j, y) \in I_{j,R}Y : u_j \in A\}, & \text{for } j = 1, 2, \\ I_RY(A) &= \{(u, y) \in I_RY : u \in A\}. \end{aligned}$$

We will always use the following identifications : for $R' \leq R$,

$$(3.8) \quad Z_{j,R'} \subseteq Z_{j,R}, \quad \text{for } j = 1, 2,$$

which is the unique isometric inclusion fixing $Z_{j,0}$; for $R' \leq 2R$,

$$(3.9) \quad Z_{j,R'} \subseteq Z_{j,2R} \subseteq Z_R, \quad \text{for } j = 1, 2,$$

where the second inclusion is induced by (3.5).

Let $D_{Z_R}^F$ be the Hodge-de Rham operator acting on $\Omega^\bullet(Z_R, F)$ (see §0.4), which is the central object in this section.

3.2. Models of eigenspaces associated to small eigenvalues. Let $\mathcal{H}_{L^2}^\bullet(Z_{j,\infty}, F)$ and $\mathcal{H}^\bullet(Z_{j,\infty}, F)$ ($j = 1, 2$) be as (2.34) and (2.35) with Z_∞ replaced by $Z_{j,\infty}$ and u replaced by u_j (cf. (3.1)). It is important to notice that $\frac{\partial}{\partial u_2}$ points to the inner side of Z_2 , which is different from the choice in (2.35). Set

$$(3.10) \quad \mathcal{H}^\bullet(Z_{12,\infty}, F) = \left\{ (\omega_1, \omega_2, \hat{\omega}) : (\omega_1, \hat{\omega}) \in \mathcal{H}^\bullet(Z_{1,\infty}, F), (\omega_2, \hat{\omega}) \in \mathcal{H}^\bullet(Z_{2,\infty}, F) \right\}.$$

Let $\mathcal{L}_j^\bullet \subseteq \mathcal{H}^\bullet(Y, F[du])$ ($j = 1, 2$) be the set of limiting values of $\mathcal{H}^\bullet(Z_{j,\infty}, F)$ (cf. (2.43)). There are natural injection

$$(3.11) \quad \begin{aligned} \mathcal{H}_{L^2}^\bullet(Z_{1,\infty}, F) \oplus \mathcal{H}_{L^2}^\bullet(Z_{2,\infty}, F) &\rightarrow \mathcal{H}^\bullet(Z_{12,\infty}, F) \\ (\omega_1, \omega_2) &\mapsto (\omega_1, \omega_2, 0), \end{aligned}$$

and surjection

$$(3.12) \quad \begin{aligned} \mathcal{H}^\bullet(Z_{12,\infty}, F) &\rightarrow \mathcal{L}_1^\bullet \cap \mathcal{L}_2^\bullet \\ (\omega_1, \omega_2, \hat{\omega}) &\mapsto \hat{\omega}, \end{aligned}$$

which induce the following short exact sequence

$$(3.13) \quad 0 \rightarrow \mathcal{H}_{L^2}^\bullet(Z_{1,\infty}, F) \oplus \mathcal{H}_{L^2}^\bullet(Z_{2,\infty}, F) \rightarrow \mathcal{H}^\bullet(Z_{12,\infty}, F) \rightarrow \mathcal{L}_1^\bullet \cap \mathcal{L}_2^\bullet \rightarrow 0.$$

Recall that the L^2 -norm $\|\cdot\|$ is defined in §0.4. For any $(\omega_1, \omega_2, \hat{\omega}) \in \mathcal{H}^\bullet(Z_{12,\infty}, F)$, set

$$(3.14) \quad \|(\omega_1, \omega_2, \hat{\omega})\|_{\mathcal{H}^\bullet(Z_{12,\infty}, F), R}^2 = \|\omega_1\|_{Z_{1,R}}^2 + \|\omega_2\|_{Z_{2,R}}^2.$$

We will drop the subscript R , if $R = 0$. By (2.20) and (2.35), there exists $C > 0$, such that, for any $(\omega_1, \omega_2, \hat{\omega}) \in \mathcal{H}^\bullet(Z_{12,\infty}, F)$,

$$(3.15) \quad \|(\omega_1, \omega_2, \hat{\omega})\|_{\mathcal{H}^\bullet(Z_{12,\infty}, F), R}^2 \leq (1 + CR) \|(\omega_1, \omega_2, \hat{\omega})\|_{\mathcal{H}^\bullet(Z_{12,\infty}, F)}^2.$$

In the rest of this section, $\mathcal{H}^\bullet(Z_{12,\infty}, F)$ will serve as the model space of $\ker(D_{Z_R}^{F,2})$.

Recall that δ_Y was defined in (2.12). For any $\lambda \in]-\delta_Y, 0[\cup]0, \delta_Y[$, $j = 1, 2$, set

$$(3.16) \quad \begin{aligned} \mathcal{E}_\lambda(Z_{j,\infty}, F) &= \left\{ (\omega, \omega^{\text{zm}}) : \omega \in \Omega^\bullet(Z_{j,\infty}, F) \text{ is a generalized eigensection of } D_{Z_{j,\infty}}^F \right. \\ &\quad \left. \text{with eigenvalue } \lambda, \omega^{\text{zm}} \in \Omega^\bullet(I_{j,\infty}Y, F) \text{ is the zeromode of } \omega \right\}. \end{aligned}$$

Recall that f_R is defined in (3.4). For any $R > 0$, set

$$(3.17) \quad \begin{aligned} \mathcal{E}_{\lambda,R}(Z_{12,\infty}, F) &= \left\{ (\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}}) : (\omega_j, \omega_j^{\text{zm}}) \in \mathcal{E}_\lambda(Z_{j,\infty}, F), \text{ for } j = 1, 2, \right. \\ &\quad \left. \omega_1^{\text{zm}}|_{I_{1,\infty}Y([0,2R])} = f_R^*(\omega_2^{\text{zm}}|_{I_{2,\infty}Y([-2R,0]))} \right\}. \end{aligned}$$

Let $C_j(\lambda) \in \text{End}(\mathcal{H}^\bullet(Y, F[du]))$ ($j = 1, 2$) be the scattering matrices associated to $D_{Z_{j,\infty}}^F$. For technical reasons, we take the following definition of scattering matrix : $C_j(\lambda)$ is the unique matrix such that (2.31) holds with u replaced by u_j (cf. (3.1)). Since $\frac{\partial}{\partial u_2}$ points to the inner side of Z_2 , which opposites the choice in §2.3, $C_2(\lambda)$ is the inverse of the scattering matrix in the sense of Definition 2.4. Set

$$(3.18) \quad C_{12}(\lambda) = C_2^{-1}(\lambda)C_1(\lambda) \in \text{End}(\mathcal{H}^\bullet(Y, F[du])) .$$

For any $R \geq 0$, set

$$(3.19) \quad \Lambda_R = \left\{ \lambda \in \mathbb{R} : \det \left(e^{4i\lambda R} C_{12}(\lambda) \big|_{\mathcal{H}^\bullet(Y, F)} - 1 \right) = 0 \right\}$$

(counting multiplicity). By (2.31), (3.4) and (3.16), we have

$$(3.20) \quad \left\{ \lambda \in \mathbb{R} : \mathcal{E}_{\lambda, R}(Z_{12, \infty}, F) \neq \{0\} \right\} = \Lambda_R .$$

For any $A \subseteq]-\delta_Y, 0[\cup]0, \delta_Y[$, set

$$(3.21) \quad \mathcal{E}_{A, R}(Z_{12, \infty}, F) = \bigoplus_{\lambda \in A} \mathcal{E}_{\lambda, R}(Z_{12, \infty}, F) .$$

For any $(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}}) \in \mathcal{E}_{A, R}(Z_{12, \infty}, F)$, set

$$(3.22) \quad \|(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}})\|_{\mathcal{E}_{A, R}(Z_{12, \infty}, F)}^2 = \|\omega_1\|_{Z_{1,0}}^2 + \|\omega_2\|_{Z_{2,0}}^2 .$$

In the rest of this section, $\mathcal{E}_{A, R}(Z_{12, \infty}, F)$ will serve as the model space of the eigenspace of $D_{Z_R}^F$ with eigenvalues in A .

3.3. Approximating the kernels. Let $\gamma \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\gamma \geq 0$, $\text{supp}(\gamma) \subseteq [-\frac{1}{2}, \frac{1}{2}]$ and $\int_{-\frac{1}{2}}^{\frac{1}{2}} \gamma(s) ds = 1$. We define $\chi_{2,1} \in \mathcal{C}^\infty([-1, 1])$ by

$$(3.23) \quad \chi_{2,1}(u) = \begin{cases} 0 & \text{if } -1 \leq u < 0 , \\ \int_{-1}^{2u-1} \gamma(s) ds & \text{if } 0 \leq u \leq 1 . \end{cases}$$

Then $\chi_{2,1}(u) = 1$ for $u > \frac{3}{4}$. For $j = 1, 2$, we define $\chi_{j,R} \in \mathcal{C}^\infty([-R, R])$ by

$$(3.24) \quad \chi_{j,R}(u) = \chi_{2,1}((-1)^j u / R) .$$

Then $\chi_{j,R}$ can be seen as a function on $I_R Y$, i.e., for $(u, y) \in I_R Y$, $\chi_{j,R}(u, y) = \chi_{j,R}(u)$.

We recall that the following maps are defined in Definition 2.10,

$$(3.25) \quad \mathcal{R}_{d^F}, \mathcal{R}_{d^F, *} : \mathcal{H}^\bullet(Z_{j, \infty}, F) \rightarrow \Omega^\bullet(I_{j, \infty} Y, F) , \quad \text{for } j = 1, 2 .$$

Composing the identification $I_R Y \simeq I_{j, 2R} Y$ ($j = 1, 2$) induced by (3.9) and the injection $I_{j, 2R} Y \subseteq I_{j, \infty} Y$ induced by (3.8), we get $I_R Y \hookrightarrow I_{j, \infty} Y$, which induces

$$(3.26) \quad \Omega^\bullet(I_{j, \infty} Y, F) \rightarrow \Omega^\bullet(I_R Y, F) .$$

Composing (3.25) and (3.26), we get

$$(3.27) \quad \mathcal{R}_{d^F, j}, \mathcal{R}_{d^F, *, j} : \mathcal{H}^\bullet(Z_{j, \infty}, F) \rightarrow \Omega^\bullet(I_R Y, F) , \quad \text{for } j = 1, 2 .$$

Definition 3.1. We define

$$(3.28) \quad F_{Z_R}, G_{Z_R} : \mathcal{H}^\bullet(Z_{12,\infty}, F) \rightarrow \Omega^\bullet(Z_R, F)$$

by, for $(\omega_1, \omega_2, \hat{\omega}) \in \mathcal{H}^\bullet(Z_{12,\infty}, F)$,

$$(3.29) \quad \begin{aligned} F_{Z_R}(\omega_1, \omega_2, \hat{\omega})|_{Z_{j,0}} &= G_{Z_R}(\omega_1, \omega_2, \hat{\omega})|_{Z_{j,0}} = \omega_j, \quad \text{for } j = 1, 2, \\ F_{Z_R}(\omega_1, \omega_2, \hat{\omega})|_{I_R Y} &= \pi_Y^* \hat{\omega} + \sum_{j=1}^2 d^F \left(\chi_{j,R} \mathcal{R}_{d^F,j}(\omega_j, \hat{\omega}) \right), \\ G_{Z_R}(\omega_1, \omega_2, \hat{\omega})|_{I_R Y} &= \pi_Y^* \hat{\omega} + \sum_{j=1}^2 d^{F,*} \left(\chi_{j,R} \mathcal{R}_{d^{F,*},j}(\omega_j, \hat{\omega}) \right). \end{aligned}$$

By (2.42), F_{Z_R} and G_{Z_R} are well-defined. Furthermore, we have

$$(3.30) \quad d^F F_{Z_R}(\omega_1, \omega_2, \hat{\omega}) = d^{F,*} G_{Z_R}(\omega_1, \omega_2, \hat{\omega}) = 0.$$

Remark 3.2. Such gluing technique is initiated by Atiyah-Patodi-Singer [1]. They glue ω_1 and ω_2 directly using partitions of unity. The difference between the standard Atiyah-Patodi-Singer gluing and ours is $\mathcal{O}(e^{-cR})$ -small as $R \rightarrow \infty$.

We recall that $U_j \subseteq Z_j$ ($j = 1, 2$) is a neighborhood of $Y = \partial Z_j$. Gluing the identifications $U_1 =]-1, 0] \times Y$, $I_R Y = [-R, R] \times Y$, $U_2 = [0, 1[\times Y$ by shifting the coordinates, we get the identification $U_1 \cup I_R Y \cup U_2 =]-R-1, R+1[\times Y$. Let $\phi_R :]-R-1, R+1[\rightarrow]-1, 1[$ be a smooth function such that

$$(3.31) \quad \phi(-u) = -\phi(u), \phi'(u) > 0, \phi_R(u) = u + R \text{ for } u \in [-R-1, -R-1/2].$$

We define the diffeomorphism $\varphi_R : Z_R \rightarrow Z$, such that

$$(3.32) \quad \begin{aligned} \varphi_R|_{Z_j \setminus U_j} &= \text{Id}_{Z_j \setminus U_j}, \quad \text{for } j = 1, 2, \\ \varphi_R(u, y) &= (\phi_R(u), y) \in U_1 \cup U_2 \subseteq Z \quad \text{for } (u, y) \in U_1 \cup I_R Y \cup U_2 \subseteq Z_R. \end{aligned}$$

Then φ_R induces the canonical isomorphism $H^\bullet(Z_R, Y) \simeq H^\bullet(Z, F)$.

Proposition 3.3. For $R' > R > 1$, $(\omega_1, \omega_2, \hat{\omega}) \in \mathcal{H}^\bullet(Z_{12,\infty}, F)$ with $\hat{\omega} \in \mathcal{H}^\bullet(Y, F)$,

$$(3.33) \quad [F_{Z_R}(\omega_1, \omega_2, \hat{\omega})] = [F_{Z_{R'}}(\omega_1, \omega_2, \hat{\omega})] \in H^\bullet(Z, F).$$

Proof. By inserting enough numbers between R and R' , we may assume that $R'/R \leq 7/6$.

We define $\tilde{\phi}_{R,R'} : [-R, R] \rightarrow [-R', R']$ by

$$(3.34) \quad \tilde{\phi}_{R,R'}(u) = \begin{cases} u - R' + R & \text{if } u \in [-R, -\frac{1}{8}R], \\ u - (R' - R)\chi_{1,R/8}(u) & \text{if } u \in [-\frac{1}{8}R, 0], \\ u + (R' - R)\chi_{2,R/8}(u) & \text{if } u \in [0, \frac{1}{8}R], \\ u + R' - R & \text{if } u \in [\frac{1}{8}R, R]. \end{cases}$$

We construct a diffeomorphism $\tilde{\varphi}_{R,R'} : Z_R \rightarrow Z_{R'}$, such that, the restriction of $\tilde{\varphi}_{R,R'}$ to $Z_{1,0} \cup Z_{2,0} \simeq Z_R \setminus I_R Y \simeq Z_{R'} \setminus I_{R'} Y$ is the identity map, and, for any $(u, y) \in I_R Y$, $\tilde{\varphi}_{R,R'}(u, y) = (\tilde{\phi}_{R,R'}(u), y) \in I_{R'} Y$. Then $\tilde{\varphi}_{R,R'}$ is homotopic to $\varphi_{R'}^{-1} \circ \varphi_R$.

Let $\mu \in \Omega^\bullet(Z_R, F)$, such that

$$(3.35) \quad \mu|_{Z_R \setminus I_R Y} = 0, \quad \mu|_{I_R Y} = \sum_{j=1}^2 (\chi_{j,R} - \chi_{j,R'}) \mathcal{R}_{d^F,j}(\omega_j, \hat{\omega}).$$

By (3.29), (3.34) and (3.35), we have

$$(3.36) \quad F_{Z_R}(\omega_1, \omega_2, \hat{\omega}) - \tilde{\varphi}_{R,R'}^* F_{Z_{R'}}(\omega_1, \omega_2, \hat{\omega}) = d^F \mu.$$

□

We recall that $\|\cdot\|$ is defined by (0.21) and $\|\cdot\|_{\mathcal{H}^\bullet(Z_{12,\infty},F),R}$ is defined by (3.14).

Proposition 3.4. *There exist $c > 0$, $R_0 > 0$, such that, for $R > R_0$, $(\omega_1, \omega_2, \hat{\omega}) \in \mathcal{H}^\bullet(Z_{12,\infty}, F)$, we have*

$$(3.37) \quad 1 - e^{-cR} \leq \frac{\|F_{Z_R}(\omega_1, \omega_2, \hat{\omega})\|_{Z_R}}{\|(\omega_1, \omega_2, \hat{\omega})\|_{\mathcal{H}^\bullet(Z_{12,\infty},F),R}} \leq 1 + e^{-cR}.$$

Proof. It is sufficient to show that

$$(3.38) \quad \|F_{Z_R}(\omega_1, \omega_2, \hat{\omega}) - \omega_j\|_{Z_{j,R}} \leq e^{-cR} \|\omega_1\|_{Z_{j,0}}, \quad \text{for } j = 1, 2.$$

Because of the symmetry, we will only show the case $j = 1$.

By the construction, $F_{Z_R}(\omega_1, \omega_2, \hat{\omega}) - \omega_1$ is zero on $Z_{1,0}$. On $I_{1,R}Y$, by (2.42), we have

$$(3.39) \quad \begin{aligned} F_{Z_R}(\omega_1, \omega_2, \hat{\omega}) - \omega_1 &= d^F(\chi_{1,R} \mathcal{R}_{d^F,1}(\omega_1, \hat{\omega})) + \pi_Y^* \hat{\omega} - \omega_1 \\ &= \left(\frac{\partial}{\partial u} \chi_{1,R} \right) du \wedge \mathcal{R}_{d^F,1}(\omega_1, \hat{\omega}) + (\chi_{1,R} - 1)(\omega_1 - \pi_Y^* \hat{\omega}). \end{aligned}$$

By the construction of $\chi_{1,R}$, $\frac{\partial}{\partial u} \chi_{1,R}$ is bounded by 1 and with support in $I_R Y([- \frac{3}{4}R, -\frac{1}{4}R])$; $\chi_{1,R} - 1$ is bounded by 1 and with support in $I_R Y([- \frac{3}{4}R, 0])$. Then

$$(3.40) \quad \|F_{Z_R}(\omega_1, \omega_2, \hat{\omega}) - \omega_1\|_{Z_{1,R}} \leq \|\mathcal{R}_{d^F,1}(\omega_1, \hat{\omega})\|_{I_R Y([- \frac{3}{4}R, -\frac{1}{4}R])} + \|\omega_1 - \pi_Y^* \hat{\omega}\|_{I_R Y([- \frac{3}{4}R, 0])}.$$

By Definition 2.10, we have

$$(3.41) \quad \|\mathcal{R}_{d^F,1}(\omega_1, \hat{\omega})\|_{I_R Y([- \frac{3}{4}R, -\frac{1}{4}R])}^2 \leq \delta_Y^{-2} e^{-\frac{1}{2}\delta_Y R} \|\omega_1\|_{\partial Z_{1,0}}^2.$$

By Lemma 2.1, (2.18) and (2.19), we have

$$(3.42) \quad \begin{aligned} \|\omega_1 - \pi_Y^* \hat{\omega}\|_{I_R Y([- \frac{3}{4}R, 0])}^2 &\leq \left(\frac{1}{1 - e^{-\frac{3}{4}\delta_Y R}} \right)^2 \cdot \delta_Y^{-1} \cdot \|\omega_1 - \pi_Y^* \hat{\omega}\|_{\partial Z_{1,R/4} \cup \partial Z_{1,R}}^2 \\ &\leq \left(\frac{1}{1 - e^{-\frac{3}{4}\delta_Y R}} \right)^2 \cdot 2\delta_Y^{-1} e^{-\frac{1}{2}\delta_Y R} \|\omega_1\|_{\partial Z_{1,0}}^2. \end{aligned}$$

Comparing (3.40)-(3.42), it only rests to show that

$$(3.43) \quad \|\omega_1\|_{\partial Z_{1,0}} \leq C \|\omega_1\|_{Z_{1,0}}.$$

Let $\|\cdot\|_{1,Z_{1,0}}$ be the H^1 -norm on $\mathcal{C}^\infty(Z_{1,0}, F)$. We fix $\varepsilon > 0$. Because of the ellipticity of Hodge-de Rham operator, we may suppose that, for any $\omega \in \Omega^\bullet(Z_{1,\infty}, F)$,

$$(3.44) \quad \|\omega\|_{1,Z_{1,0}}^2 \leq \|\omega\|_{Z_{1,\varepsilon}}^2 + \|D_{Z_{1,\infty}}^F \omega\|_{Z_{1,\varepsilon}}^2.$$

In particular,

$$(3.45) \quad \|\omega_1\|_{1,Z_{1,0}}^2 \leq \|\omega_1\|_{Z_{1,\varepsilon}}^2 .$$

By trace theorem, there exists $C_2 > 0$, such that, for any ω_1 , we have

$$(3.46) \quad \|\omega_1\|_{\partial Z_{1,0}}^2 \leq C_2 \|\omega_1\|_{1,Z_{1,0}}^2 .$$

By (3.15), (3.45) and (3.46), we get (3.43). \square

For going further, we need a uniform Sobolev inequality on all Z_R ($R \geq 0$). Let $m \in \mathbb{N}$ such that $m > \frac{1}{2} \dim Z_R$. We recall that $\|\cdot\|_{\mathcal{C}^0}$ is defined by (0.22).

Proposition 3.5. *There exists $C > 0$, such that, for $R > 0$, $\omega \in \Omega^\bullet(Z_R, F)$, we have*

$$(3.47) \quad \|\omega\|_{\mathcal{C}^0, Z_R} \leq C \left(\|\omega\|_{Z_R} + \|D_{Z_R}^{F,m} \omega\|_{Z_R} \right) .$$

Proof. One may repeat the proof of the classical Sobolev inequality on each Z_R and find that the constants C , which, a priori, depend on R , are uniformly bounded above. \square

Let $P^{\ker(D_{Z_R}^{F,2})} : \Omega^\bullet(Z_R, F) \rightarrow \ker(D_{Z_R}^{F,2})$ be the orthogonal projection.

Definition 3.6. Set

$$(3.48) \quad \mathcal{F}_{Z_R} = P^{\ker(D_{Z_R}^{F,2})} \circ F_{Z_R} , \quad \mathcal{G}_{Z_R} = P^{\ker(D_{Z_R}^{F,2})} \circ G_{Z_R} .$$

Proposition 3.7. *There exist $c > 0$, $R_0 > 0$, such that, for $R > R_0$, $(\omega_1, \omega_2, \hat{\omega}) \in \mathcal{H}^\bullet(Z_{12,\infty}, F)$, we have*

$$(3.49) \quad \|(F_{Z_R} - \mathcal{F}_{Z_R})(\omega_1, \omega_2, \hat{\omega})\|_{\mathcal{C}^0, Z_R} \leq e^{-cR} \|(\omega_1, \omega_2, \hat{\omega})\|_{\mathcal{H}^\bullet(Z_{12,\infty}, F)} .$$

As a consequence $\mathcal{F}_{Z_R} : \mathcal{H}^\bullet(Z_{12,\infty}, F) \rightarrow \ker(D_{Z_R}^{F,2})$ is injective for R large enough.

Proof. By (2.42) and (3.29), $\text{supp}((F_{Z_R} - G_{Z_R})(\omega_1, \omega_2, \hat{\omega})) \subseteq I_R Y$, and

$$(3.50) \quad (F_{Z_R} - G_{Z_R})(\omega_1, \omega_2, \hat{\omega})|_{I_R Y} = \sum_{j=1}^2 \left(\frac{\partial}{\partial u} \chi_{j,R} \right) \left(du \wedge \mathcal{R}_{d^F,j}(\omega_j, \hat{\omega}) + i \frac{\partial}{\partial u} \mathcal{R}_{d^F,*,j}(\omega_j, \hat{\omega}) \right) .$$

More generally, by (1.6), (2.42) and (3.50), for any $m \in \mathbb{N}$,

$$(3.51) \quad \begin{aligned} & D_{Z_R}^{F,2m} (F_{Z_R} - G_{Z_R})(\omega_1, \omega_2, \hat{\omega})|_{I_R Y} \\ &= (-1)^m \sum_{j=1}^2 \left(\frac{\partial^{2m+1}}{\partial u^{2m+1}} \chi_{j,R} \right) \left(du \wedge \mathcal{R}_{d^F,j}(\omega_j, \hat{\omega}) + i \frac{\partial}{\partial u} \mathcal{R}_{d^F,*,j}(\omega_j, \hat{\omega}) \right) . \end{aligned}$$

Set

$$(3.52) \quad \alpha_m = \sup_{u \in [-1,1]} \left| \frac{\partial^m}{\partial u^m} \chi_{2,1}(u) \right| .$$

Since $\text{supp}(\frac{\partial}{\partial u}\chi_{1,R}) \subseteq [-\frac{3}{4}R, -\frac{1}{4}R]$ and $\text{supp}(\frac{\partial}{\partial u}\chi_{2,R}) \subseteq [\frac{1}{4}R, \frac{3}{4}R]$, we get

(3.53)

$$\begin{aligned} & \left\| D_{Z_R}^{F,2m} (F_{Z_R} - G_{Z_R}) (\omega_1, \omega_2, \hat{\omega}) \right\|_{Z_R}^2 \\ & \leq \alpha_{2m+1}^2 R^{-4m-2} \left\| \mathcal{R}_{d^F,1}(\omega_1, \hat{\omega}) \right\|_{I_{RY}[-\frac{3}{4}R, -\frac{1}{4}R]}^2 + \alpha_{2m+1}^2 R^{-4m-2} \left\| \mathcal{R}_{d^F,2}(\omega_2, \hat{\omega}) \right\|_{I_{RY}[\frac{1}{4}R, \frac{3}{4}R]}^2 \\ & + \alpha_{2m+1}^2 R^{-4m-2} \left\| \mathcal{R}_{d^{F,*},1}(\omega_1, \hat{\omega}) \right\|_{I_{RY}[-\frac{3}{4}R, -\frac{1}{4}R]}^2 + \alpha_{2m+1}^2 R^{-4m-2} \left\| \mathcal{R}_{d^{F,*},2}(\omega_2, \hat{\omega}) \right\|_{I_{RY}[\frac{1}{4}R, \frac{3}{4}R]}^2 . \end{aligned}$$

By Definition 2.10, we have

(3.54)

$$\begin{aligned} & \left\| \mathcal{R}_{d^F,1}(\omega_1, \hat{\omega}) \right\|_{I_{RY}[-\frac{3}{4}R, -\frac{1}{4}R]}^2 + \left\| \mathcal{R}_{d^{F,*},1}(\omega_1, \hat{\omega}) \right\|_{I_{RY}[-\frac{3}{4}R, -\frac{1}{4}R]}^2 \leq 2\delta_Y^{-2} e^{-\frac{1}{2}\delta_Y R} \|\omega_1\|_{\partial Z_{1,0}}^2 , \\ & \left\| \mathcal{R}_{d^F,2}(\omega_2, \hat{\omega}) \right\|_{I_{RY}[\frac{1}{4}R, \frac{3}{4}R]}^2 + \left\| \mathcal{R}_{d^{F,*},2}(\omega_2, \hat{\omega}) \right\|_{I_{RY}[\frac{1}{4}R, \frac{3}{4}R]}^2 \leq 2\delta_Y^{-2} e^{-\frac{1}{2}\delta_Y R} \|\omega_2\|_{\partial Z_{2,0}}^2 . \end{aligned}$$

By (3.53) and (3.54), we have

(3.55)

$$\left\| D_{Z_R}^{F,2m} (F_{Z_R} - G_{Z_R}) (\omega_1, \omega_2, \hat{\omega}) \right\|_{Z_R}^2 \leq \alpha_{2m+1}^2 \delta_Y^{-2} R^{-4m-2} e^{-\frac{1}{2}\delta_Y R} (\|\omega_1\|_{\partial Z_{1,0}}^2 + \|\omega_2\|_{\partial Z_{2,0}}^2) .$$

Same as (3.43), we have

$$(3.56) \quad \|\omega_1\|_{\partial Z_{1,0}}^2 + \|\omega_2\|_{\partial Z_{2,0}}^2 \leq C(\|\omega_1\|_{Z_{1,0}}^2 + \|\omega_2\|_{Z_{2,0}}^2) = C\|(\omega_1, \omega_2, \hat{\omega})\|_{\mathcal{H}^\bullet(Z_{12,\infty}, F)}^2 .$$

By (3.15), (3.55) and (3.56), for any $m \in \mathbb{N}$, there exist $c_m > 0$, $R_m > 0$, such that, for any $R > R_m$, any $(\omega_1, \omega_2, \hat{\omega}) \in \mathcal{H}^\bullet(Z_{12,\infty}, F)$, we have

$$(3.57) \quad \left\| D_{Z_R}^{F,2m} (F_{Z_R} - G_{Z_R}) (\omega_1, \omega_2, \hat{\omega}) \right\|_{Z_R} \leq e^{-c_m R} \|(\omega_1, \omega_2, \hat{\omega})\|_{\mathcal{H}^\bullet(Z_{12,\infty}, F)} .$$

Set

$$(3.58) \quad \begin{aligned} \mu_0 &= \mathcal{F}_{Z_R}(\omega_1, \omega_2, \hat{\omega}) - \mathcal{G}_{Z_R}(\omega_1, \omega_2, \hat{\omega}) , \\ \mu_1 &= F_{Z_R}(\omega_1, \omega_2, \hat{\omega}) - \mathcal{F}_{Z_R}(\omega_1, \omega_2, \hat{\omega}) , \\ \mu_2 &= G_{Z_R}(\omega_1, \omega_2, \hat{\omega}) - \mathcal{G}_{Z_R}(\omega_1, \omega_2, \hat{\omega}) , \end{aligned}$$

then

$$(3.59) \quad (F_{Z_R} - G_{Z_R})(\omega_1, \omega_2, \hat{\omega}) = \mu_0 + \mu_1 - \mu_2 .$$

By Theorem 1.1 and (3.30), we have

$$(3.60) \quad \mu_0 \in \ker \left(D_{Z_R}^{F,2} \right) , \quad \mu_1 \in \text{Im}(d^F) , \quad \mu_2 \in \text{Im}(d^{F,*}) .$$

For $m > 0$, by (1.6), $D_{Z_R}^{F,2m}$ commutes with d^F and $d^{F,*}$, thus

$$(3.61) \quad D_{Z_R}^{F,2m} \mu_1 \in \text{Im}(d^F) , \quad D_{Z_R}^{F,2m} \mu_2 \in \text{Im}(d^{F,*}) .$$

As a consequence, $D_{Z_R}^{F,2m} \mu_0$, $D_{Z_R}^{F,2m} \mu_1$ and $D_{Z_R}^{F,2m} \mu_2$ are mutually orthogonal. Thus, for $m \in \mathbb{N}$, by (3.59) and (3.61), we get

(3.62)

$$\left\| D_{Z_R}^{F,2m} (F_{Z_R} - \mathcal{F}_{Z_R})(\omega_1, \omega_2, \hat{\omega}) \right\|_{Z_R} = \left\| D_{Z_R}^{F,2m} \mu_1 \right\|_{Z_R} \leq \left\| D_{Z_R}^{F,2m} (F_{Z_R} - G_{Z_R})(\omega_1, \omega_2, \hat{\omega}) \right\|_{Z_R} .$$

By (3.57) and (3.62), we get

$$(3.63) \quad \left\| D_{Z_R}^{F,2m}(F_{Z_R} - \mathcal{F}_{Z_R})(\omega_1, \omega_2, \hat{\omega}) \right\|_{Z_R} \leq e^{-cR} \|(\omega_1, \omega_2, \hat{\omega})\|_{\mathcal{H}^\bullet(Z_{12,\infty}, F)}.$$

By (3.63) and Proposition 3.5, we get (3.49).

And the injectivity of \mathcal{F}_{Z_R} follows from (3.37) and (3.49). \square

Remark 3.8. The Hodge decomposition is used in an essential way at (3.60), thus, the proof of Proposition 3.7 cannot be applied to general Dirac operators.

Proposition 3.9. *For any $\varepsilon > 0$, there exists $R_0 > 0$ such that, for $R > R_0$, any eigensection of $D_{Z_R}^F$ with eigenvalue $\lambda \in] - R^{-1-\varepsilon}, R^{-1-\varepsilon}[$ is contained in the image of \mathcal{F}_{Z_R} .*

Proof. Suppose the contrary, i.e., there exist $R_i \rightarrow +\infty$, $\omega_i \in \Omega^\bullet(Z_{R_i}, F)$ and $\lambda_i \in] - R_i^{-1-\varepsilon}, R_i^{-1-\varepsilon}[$, such that

$$(3.64) \quad \omega_i \neq 0, \quad D_{Z_{R_i}}^F \omega_i = \lambda_i \omega_i,$$

$$(3.65) \quad \omega_i \perp \text{Im}(\mathcal{F}_{Z_{R_i}}).$$

By Lemma 2.1, we may multiply a suitable constant, such that

$$(3.66) \quad \|\omega_i\|_{Z_{R_i} \setminus I_{R_i} Y}^2 = \|\omega_i\|_{Z_{1,0}}^2 + \|\omega_i\|_{Z_{2,0}}^2 = 1.$$

By Lemma 2.1 and (3.64), there exists $C > 0$, such that, for any $T \in \mathbb{N}$, if $R_i \geq T$,

$$(3.67) \quad \|\omega_i\|_{Z_{1,T}}^2 \leq 1 + CT.$$

Thus, for any $T \in \mathbb{N}$ fixed, the series $(\omega_i|_{Z_{1,T}})_i$ is L^2 -bounded.

Since λ_i are bounded, using Rellich's lemma, we may suppose, by extracting a subsequence, that $(\omega_i|_{Z_{1,T}})_i$ converges with respect to the k -th Sobolev norm for all $k \in \mathbb{N}$. By Sobolev imbedding theorem, the convergence is with respect to \mathcal{C}^1 -norm. By a diagonal argument (involving i and T), we get $\omega_{1,\infty} \in \Omega^\bullet(Z_{1,\infty}, F)$, such that, for any $T \in \mathbb{N}$, $(\omega_i|_{Z_{1,T}})_i$ converges to $\omega_{1,\infty}|_{Z_{1,T}}$ (with respect to \mathcal{C}^1 -norm). By taking the limit of (3.64), we get $D_{Z_{1,\infty}}^F \omega_{1,\infty} = 0$. Furthermore, by taking the limit of (3.67), we get

$$(3.68) \quad \|\omega_{1,\infty}\|_{Z_{1,T}}^2 \leq 1 + CT, \quad \text{for } T \in \mathbb{N}.$$

By (2.13) and (3.68), $\omega_{1,\infty}$ is an extended L^2 -solution, i.e., there exists $\hat{\omega}_1$ such that $(\omega_{1,\infty}, \hat{\omega}_1) \in \mathcal{H}^\bullet(Z_{1,\infty}, F)$. In particular,

$$(3.69) \quad \omega_i^{\text{zm}}|_{\partial Z_{1,0}} \rightarrow \hat{\omega}_1, \quad \text{as } i \rightarrow \infty.$$

By repeating the same procedure for $\omega_i|_{Z_{2,T}}$, we find $(\omega_{2,\infty}, \hat{\omega}_2) \in \mathcal{H}^\bullet(Z_{2,\infty}, F)$ satisfying the same properties. In particular,

$$(3.70) \quad \omega_i^{\text{zm}}|_{\partial Z_{2,0}} \rightarrow \hat{\omega}_2, \quad \text{as } i \rightarrow \infty.$$

By (2.14), we have

$$(3.71) \quad \omega_i^{\text{zm}, \pm}|_{\partial Z_{2,0}} = e^{\pm 2\sqrt{-1}R_i \lambda_i} \omega_i^{\text{zm}, \pm}|_{\partial Z_{1,0}}.$$

Since $R_i \lambda_i \rightarrow 0$, by (3.69)-(3.71), we get

$$(3.72) \quad \hat{\omega}_1 = \hat{\omega}_2.$$

Then $(\omega_{1,\infty}, \omega_{2,\infty}, \hat{\omega}_1) \in \mathcal{H}^\bullet(Z_{12,\infty}, F)$.

Set

$$(3.73) \quad \tilde{\omega}_i = \mathcal{F}_{Z_{R_i}}(\omega_{1,\infty}, \omega_{2,\infty}, \hat{\omega}_1) .$$

Case 1, $\hat{\omega}_1 \neq 0$: We want to show that $\langle \omega_i, \tilde{\omega}_i \rangle \rightarrow \infty$ as $i \rightarrow \infty$, which contradicts (3.65).

We have

$$(3.74) \quad \langle \omega_i, \tilde{\omega}_i \rangle = \langle \omega_i, \tilde{\omega}_i \rangle_{Z_{R_i} \setminus I_{R_i} Y} + \langle \omega_i^{\text{nz}}, \tilde{\omega}_i^{\text{nz}} \rangle_{I_{R_i} Y} + \langle \omega_i^{\text{zm}}, \tilde{\omega}_i^{\text{zm}} \rangle_{I_{R_i} Y} .$$

By Lemma 2.1 and Proposition 3.7, $\langle \omega_i, \tilde{\omega}_i \rangle_{Z_{R_i} \setminus I_{R_i} Y}$ and $\langle \omega_i^{\text{nz}}, \tilde{\omega}_i^{\text{nz}} \rangle_{I_{R_i} Y}$ are bounded, when $i \rightarrow \infty$. Thus, it is sufficient to show that $\langle \omega_i^{\text{zm}}, \tilde{\omega}_i^{\text{zm}} \rangle_{I_{R_i} Y} \rightarrow \infty$ as $i \rightarrow \infty$.

We have

$$(3.75) \quad \langle \omega_i^{\text{zm}}, \tilde{\omega}_i^{\text{zm}} \rangle_{I_{R_i} Y} = \langle \omega_i^{\text{zm}}, \pi_Y^* \hat{\omega}_1 \rangle_{I_{R_i} Y} + \langle \omega_i^{\text{zm}}, \tilde{\omega}_i^{\text{zm}} - \pi_Y^* \hat{\omega}_1 \rangle_{I_{R_i} Y} .$$

By Definition 2.10, 3.1,

$$(3.76) \quad \pi_Y^* \hat{\omega}_1 = \left(F_{Z_{R_i}}(\omega_{1,\infty}, \omega_{2,\infty}, \hat{\omega}_1) \Big|_{I_{R_i} Y} \right)^{\text{zm}} .$$

Then, by Proposition 3.7,

$$(3.77) \quad \langle \omega_i^{\text{zm}}, \tilde{\omega}_i^{\text{zm}} - \pi_Y^* \hat{\omega}_1 \rangle_{I_{R_i} Y} \rightarrow 0, \text{ as } i \rightarrow \infty .$$

By (2.14) and the fact that $R_i \lambda_i \rightarrow 0$, the restriction of ω_i^{zm} to $I_{R_i} Y(u)$ ($u \in [-R_i, R_i]$) converges uniformly to the same limit. Then, by (3.69), they all converge to $\hat{\omega}_1$. Thus,

$$(3.78) \quad \langle \omega_i^{\text{zm}}, \pi_Y^* \hat{\omega}_1 \rangle_{I_{R_i} Y} = \int_{-R_i}^{R_i} \langle \omega_i^{\text{zm}}|_{I_{R_i} Y(u)}, \hat{\omega}_1 \rangle_Y du \rightarrow +\infty, \text{ as } i \rightarrow \infty .$$

This ends the first case.

Case 2, $\hat{\omega}_1 = 0$: We want to show that

$$(3.79) \quad \langle \omega_i, \tilde{\omega}_i \rangle \rightarrow \|\omega_{1,\infty}\|_{Z_{1,\infty}}^2 + \|\omega_{2,\infty}\|_{Z_{2,\infty}}^2 > 0, \text{ as } i \rightarrow \infty ,$$

which contradicts (3.65).

For any $T > 0$, $R_i > T$, we have

$$(3.80) \quad \langle \omega_i, \tilde{\omega}_i \rangle = \langle \omega_i, \tilde{\omega}_i \rangle_{Z_{1,T} \cup Z_{2,T}} + \langle \omega_i^{\text{nz}}, \tilde{\omega}_i^{\text{nz}} \rangle_{I_{R_i} Y([-R_i+T, R_i-T])} + \langle \omega_i^{\text{zm}}, \tilde{\omega}_i^{\text{zm}} \rangle_{I_{R_i} Y([-R_i+T, R_i-T])} .$$

By Definition 3.1 and Proposition 3.7,

$$(3.81) \quad \langle \omega_i, \tilde{\omega}_i \rangle_{Z_{1,T} \cup Z_{2,T}} \rightarrow \|\omega_{1,\infty}\|_{Z_{1,T}}^2 + \|\omega_{2,\infty}\|_{Z_{2,T}}^2, \text{ as } i \rightarrow \infty .$$

By Lemma 2.1, if $R_i > \delta_Y^{-1}$ and $\lambda_i < \frac{1}{2}\delta_Y$ (which hold for i large enough),

$$(3.82) \quad \left| \langle \omega_i^{\text{nz}}, \tilde{\omega}_i^{\text{nz}} \rangle_{I_{R_i} Y([-R_i+T, R_i-T])} \right| \leq 8\delta_Y^{-1} (\|\omega_i\|_{\partial Z_{1,T}} + \|\omega_i\|_{\partial Z_{2,T}}) (\|\tilde{\omega}_i\|_{\partial Z_{1,T}} + \|\tilde{\omega}_i\|_{\partial Z_{2,T}}) .$$

Furthermore, as $i \rightarrow \infty$,

$$(3.83) \quad (\|\omega_i\|_{\partial Z_{1,T}} + \|\omega_i\|_{\partial Z_{2,T}}) (\|\tilde{\omega}_i\|_{\partial Z_{1,T}} + \|\tilde{\omega}_i\|_{\partial Z_{2,T}}) \rightarrow (\|\omega_{1,\infty}\|_{\partial Z_{1,T}} + \|\omega_{2,\infty}\|_{\partial Z_{2,T}})^2 ,$$

$$(3.84) \quad (\|\omega_{1,\infty}\|_{\partial Z_{1,T}} + \|\omega_{2,\infty}\|_{\partial Z_{2,T}})^2 \leq e^{-2\delta_Y T} (\|\omega_{1,\infty}\|_{\partial Z_{1,0}} + \|\omega_{2,\infty}\|_{\partial Z_{2,0}})^2 .$$

The same procedure as (3.77) combined with the fact that $\hat{\omega}_1 = 0$ yields

$$(3.85) \quad \langle \omega_i^{\text{zm}}, \tilde{\omega}_i^{\text{zm}} \rangle_{I_{R_i} Y([-R_i+T, R_i-T])} \rightarrow 0, \text{ as } i \rightarrow \infty .$$

By (3.80)-(3.85), we get

(3.86)

$$\begin{aligned} \limsup_{i \rightarrow \infty} \langle \omega_i, \tilde{\omega}_i \rangle &\leq \|\omega_{1,\infty}\|_{Z_{1,T}}^2 + \|\omega_{2,\infty}\|_{Z_{2,T}}^2 + 8\delta_Y^{-1} e^{-2\delta_Y T} (\|\omega_{1,\infty}\|_{\partial Z_{1,0}} + \|\omega_{2,\infty}\|_{\partial Z_{2,0}})^2, \\ \liminf_{i \rightarrow \infty} \langle \omega_i, \tilde{\omega}_i \rangle &\geq \|\omega_{1,\infty}\|_{Z_{1,T}}^2 + \|\omega_{2,\infty}\|_{Z_{2,T}}^2 - 8\delta_Y^{-1} e^{-2\delta_Y T} (\|\omega_{1,\infty}\|_{\partial Z_{1,0}} + \|\omega_{2,\infty}\|_{\partial Z_{2,0}})^2. \end{aligned}$$

From (3.86) for $T \rightarrow \infty$, we get (3.79). \square

Theorem 3.10. *There exists $R_0 > 0$ such that, for $R > R_0$, the map $\mathcal{F}_{Z_R} : \mathcal{H}^\bullet(Z_{12,\infty}, F) \rightarrow \ker(D_{Z_R}^{F,2})$ is bijective, and*

$$(3.87) \quad \mathrm{Sp}(D_{Z_R}^F) \subseteq] -\infty, -R^{-1-\varepsilon}[\cup \{0\} \cup]R^{-1-\varepsilon}, +\infty[.$$

Proof. These are immediate consequences of Proposition 3.7, Proposition 3.9. \square

3.4. Approximating the small eigenvalues. For $j = 1, 2$, let $D_{Z_j,\infty}^F$ be the restriction of $D_{Z_j,\infty}^F$ to its p.p. spectrum, which is defined in §2.3. We fix $\delta_{Z_j} > 0$ such that $D_{Z_j,\infty}^F$ has no eigenvalue in $] -\delta_{Z_j}, \delta_{Z_j}[$ other than 0. Set $\delta = \frac{1}{2} \min\{\delta_Y, \delta_{Z_1}, \delta_{Z_2}\}$.

We recall that $\mathcal{E}_{A,R}(Z_{12,\infty}, F)$ was defined in (3.21), $I_{1,R}Y, I_{2,R}Y, I_RY$ were defined at the end of subsection 3.1 and χ_R^\pm was defined at the beginning of §3.3.

Definition 3.11. We define

$$(3.88) \quad J_{A,Z_R} : \mathcal{E}_{A,R}(Z_{12,\infty}, F) \rightarrow \Omega^\bullet(Z_R, F),$$

such that, for any $(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}}) \in \mathcal{E}_{A,R}(Z_{12,\infty}, F)$,

$$(3.89) \quad \begin{aligned} J_{A,Z_R}(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}})|_{Z_{j,0}} &= \omega_j, \quad \text{for } j = 1, 2, \\ J_{A,Z_R}(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}})|_{I_RY} &= \chi_{1,R} \omega_1|_{I_{1,2R}Y} + \chi_{2,R} \omega_2|_{I_{2,2R}Y} \\ &\quad + (1 - \chi_{1,R} - \chi_{2,R}) \omega_1^{\mathrm{zm}}|_{I_{1,2R}Y}, \end{aligned}$$

where, by the identification $I_RY \simeq I_{j,2R}Y$ ($j = 1, 2$), $\omega_j|_{I_{j,2R}Y}$ is seen as a section on I_RY , and the same for $\omega_1^{\mathrm{zm}}|_{I_{1,2R}Y}$.

Let $\mathcal{E}_B(Z_R, F) \subseteq \Omega^\bullet(Z_R, F)$ be the eigenspace of $D_{Z_R}^F$ with eigenvalues in B . Let $P_{Z_R}^B : \Omega^\bullet(Z_R, F) \rightarrow \mathcal{E}_B(Z_R, F)$ be the orthogonal projection.

Definition 3.12. Set

$$(3.90) \quad \mathcal{J}_{A,B,Z_R} = P_{Z_R}^B \circ J_{A,Z_R} : \mathcal{E}_{A,R}(Z_{12,\infty}, F) \rightarrow \mathcal{E}_B(Z_R, F).$$

For $A, B \subseteq \mathbb{R}$ and $\alpha > 0$, we note $A \subseteq_\alpha B$, if $]x - \alpha, x + \alpha[\subseteq B$ for any $x \in A$.

Proposition 3.13. *There exist $R_0 > 0$, $c > 0$, such that, for $R > R_0$, $A \subseteq_{e^{-cR}} B \subseteq] -\delta, 0[\cup]0, \delta[$, $(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}}) \in \mathcal{E}_{A,R}(Z_{12,\infty}, F)$, we have*

$$(3.91) \quad \left\| (\mathcal{J}_{A,B,Z_R} - J_{A,Z_R})(\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}}) \right\|_{\mathcal{E}_B(Z_R, F)} \leq e^{-cR} \left\| (\omega_1, \omega_1^{\mathrm{zm}}, \omega_2, \omega_2^{\mathrm{zm}}) \right\|_{\mathcal{E}_{A,R}(Z_{12,\infty}, F)}.$$

As a consequence, \mathcal{J}_{A,B,Z_R} is injective for R large enough.

Proof. It suffices to consider the case $(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}}) \in \mathcal{E}_{\lambda_0, R}^\bullet(Z_{12, \infty}, F)$ with $\lambda_0 \in A$.

The same argument as (3.57) shows that, for any $m \in \mathbb{N}$, there exist $R_m > 0$, $c_m > 0$, such that, for $R > R_m$,

$$(3.92) \quad \left\| D_{Z_R}^{F, m} (D_{Z_R}^F - \lambda_0) J_{A, Z_R}(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}}) \right\|_{Z_R}^2 \leq e^{-3c_m R} \|(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}})\|_{\mathcal{E}_{A, R}(Z_{12, \infty}, F)}^2 .$$

We decompose $J_{A, Z_R}(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}}) \in \Omega^\bullet(Z_R, F)$ by the eigensections of $D_{Z_R}^F$, i.e.,

$$(3.93) \quad J_{A, Z_R}(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}}) = \sum_{\lambda} \mu_{\lambda}$$

with $D_{Z_R}^F \mu_{\lambda} = \lambda \mu_{\lambda}$, in particular, these μ_{λ} are mutually orthogonal. Then

$$(3.94) \quad \mathcal{J}_{A, B, Z_R}(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}}) = \sum_{\lambda \in B} \mu_{\lambda} .$$

By (3.92) and (3.93), we have

$$(3.95) \quad \sum_{|\lambda - \lambda_0| > e^{-c_m R}} \left\| D_{Z_R}^{F, m} \mu_{\lambda} \right\|_{Z_R}^2 \leq e^{-2c_m R} \|(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}})\|_{\mathcal{E}_{A, R}(Z_{12, \infty}, F)}^2 .$$

By Proposition 3.5 and (3.93)-(3.95), we get (3.91). \square

Lemma 3.14. *For any $\varepsilon > 0$, there exist $R_0 > 0$, $C > 0$, such that for any $R > R_0$, $\omega \in \Omega^\bullet(Z_R, F)$ an eigensection with eigenvalue $\lambda \in]-\delta + \varepsilon, 0[\cup]0, \delta - \varepsilon[$, we have*

$$(3.96) \quad \|\omega^{\text{zm}, +}\|_Y^2 + \|\omega^{\text{zm}, -}\|_Y^2 \geq C \|\omega\|_{Z_{1,0} \cup Z_{2,0}}^2 .$$

In particular, ω^{zm} is non zero.

Proof. Suppose the contrary, i.e., there exist $R_i \rightarrow +\infty$, $\omega_i \in \Omega^\bullet(Z_{R_i}, F)$ and $\lambda_i \in]-\delta + \varepsilon, 0[\cup]0, \delta - \varepsilon[$, such that

$$(3.97) \quad D_{Z_{R_i}}^F \omega_i = \lambda_i \omega_i ,$$

and

$$(3.98) \quad \frac{\|\omega_i^{\text{zm}, +}\|_Y^2 + \|\omega_i^{\text{zm}, -}\|_Y^2}{\|\omega_i\|_{Z_{1,0} \cup Z_{2,0}}^2} \rightarrow 0 , \quad \text{as } i \rightarrow \infty .$$

By extracting subsequence, we may assume that $\lambda_i \rightarrow \lambda_\infty$. By Lemma 2.1, $\|\omega_i\|_{Z_{1,0} \cup Z_{2,0}}^2 \neq 0$, we may multiply suitable constants such that

$$(3.99) \quad \|\omega_i\|_{Z_{1,0} \cup Z_{2,0}}^2 = 1 .$$

By (3.98) and (3.99), we have

$$(3.100) \quad \|\omega_i^{\text{zm}, +}\|_Y^2 + \|\omega_i^{\text{zm}, -}\|_Y^2 \rightarrow 0 , \quad \text{as } i \rightarrow \infty .$$

Same as the proof of Proposition 3.9, by extracting subsequence, we may assume that there exist $\omega_{1, \infty} \in \Omega^\bullet(Z_{1, \infty}, F)$, $\omega_{2, \infty} \in \Omega^\bullet(Z_{2, \infty}, F)$, such that, for any $T \in \mathbb{N}$, $(\omega_i|_{Z_{j, T}})_i$ converges to $\omega_{j, \infty}|_{Z_{j, T}}$ ($j = 1, 2$) with respect to the \mathcal{C}^1 -norm. By taking the limit of (3.97) and (3.100), we get, for $j = 1, 2$,

$$(3.101) \quad D_{Z_{j, \infty}}^F \omega_{j, \infty} = \lambda_\infty \omega_{j, \infty} , \quad \omega_{j, \infty}^{\text{zm}, \pm} = 0 .$$

By taking the limit of (3.99), we get

$$(3.102) \quad \|\omega_{1,\infty}\|_{Z_{1,0}}^2 + \|\omega_{2,\infty}\|_{Z_{2,0}}^2 = 1$$

Thus, at least one of $\omega_{1,\infty}, \omega_{2,\infty}$ is non zero. Without loss of generality, we may assume that $\omega_{1,\infty}$ is non zero. By (3.101), $\omega_{1,\infty}$ is zeromode free, then, by Lemma 2.1, $\omega_{1,\infty}$ is a L^2 -eigensection, so $\lambda_\infty \in \text{Sp} \left(D_{Z_{1,\infty}, \text{pp}}^F \right)$. But $|\lambda_\infty| < \delta \leq \delta_{Z_1}$, by the definition of δ_{Z_1} , we must have $\lambda_\infty = 0$. Thus, $\omega_{1,\infty} \in \mathcal{H}_{L^2}^\bullet(Z_{1,\infty}, F)$.

Recall that $\mathcal{F}_{Z_{R_i}}(\cdot, \cdot, \cdot)$ was defined in (3.48). Same as (3.79), we can show that

$$(3.103) \quad \langle \omega_i, \mathcal{F}_{Z_{R_i}}(\omega_{1,\infty}, 0, 0) \rangle \rightarrow \|\omega_{1,\infty}\|^2 > 0, \quad \text{as } i \rightarrow \infty.$$

But, by (3.97), $\lambda_i \neq 0$ and $\mathcal{F}_{Z_{R_i}}(\omega_{1,\infty}, 0, 0) \in \ker \left(D_{Z_{R_i}}^{F,2} \right)$, we have $\omega_i \perp \mathcal{F}_{Z_{R_i}}(\omega_{1,\infty}, 0, 0)$. This contradicts (3.103). \square

Lemma 3.15. *For any $\varepsilon > 0$, there exist $R_0 > 0$, $c > 0$, such that for $R > R_0$, $\omega \in \Omega^\bullet(Z_R, F)$ eigensection of $D_{Z_R}^F$ with eigenvalue $\lambda \in]-\delta + \varepsilon, 0[\cup]0, \delta - \varepsilon[$, we have*

$$(3.104) \quad \|C_j(\lambda)\omega^{\text{zm},-}|_{\partial Z_{j,0}} - \omega^{\text{zm},+}|_{\partial Z_{j,0}}\|_Y \leq e^{-cR}\|\omega\|_{Z_{1,0} \cup Z_{2,0}}, \quad \text{for } j = 1, 2.$$

In particular,

$$(3.105) \quad \|(e^{4i\lambda R}C_{12}(\lambda) - 1)\omega^{\text{zm},-}|_{\partial Z_{1,0}}\|_Y \leq e^{-cR}\|\omega\|_{Z_{1,0} \cup Z_{2,0}}.$$

Proof. We follow the argument of [31, 33].

By the symmetry, it is sufficient to establish (3.104) with $j = 1$.

Let $\omega \in \Omega^\bullet(Z_R, F)$ be an eigensection of $D_{Z_R}^F$ with eigenvalue $\lambda \in]-\delta + \varepsilon, 0[\cup]0, \delta - \varepsilon[$. Recall (2.13), there exist $\phi, \phi' \in \mathcal{H}^\bullet(Y, F)$, such that

$$(3.106) \quad \omega|_{I_{1,R}Y} = e^{-i\lambda u_1}(\phi - ic(\frac{\partial}{\partial u})\phi) + e^{i\lambda u_1}(\phi' + ic(\frac{\partial}{\partial u})\phi') + \omega^{\text{nz}}.$$

By Proposition 2.5, there exists $(\tilde{\omega}, \tilde{\omega}^{\text{zm}}) \in \mathcal{E}_\lambda^\bullet(Z_{1,\infty}, F)$ satisfying

$$(3.107) \quad \tilde{\omega}^{\text{zm}} = e^{-i\lambda u_1}(\phi - ic(\frac{\partial}{\partial u})\phi) + e^{i\lambda u_1}C_1(\lambda)(\phi - ic(\frac{\partial}{\partial u})\phi).$$

Set

$$(3.108) \quad \mu = \omega - \tilde{\omega} \in \Omega^\bullet(Z_{1,R}, F).$$

Then, μ is also an eigensection of $D_{Z_R}^F$ with eigenvalue λ . Thus

$$(3.109) \quad \langle D_{Z_R}^F \mu, \mu \rangle_{Z_{1,R}} - \langle \mu, D_{Z_R}^F \mu \rangle_{Z_{1,R}} = \langle \lambda \mu, \mu \rangle_{Z_{1,R}} - \langle \mu, \lambda \mu \rangle_{Z_{1,R}} = 0.$$

On the other hand, by (2.11) and (3.106)-(3.108), we have

$$(3.110) \quad \begin{aligned} \langle D_{Z_R}^F \mu, \mu \rangle_{Z_{1,R}} - \langle \mu, D_{Z_R}^F \mu \rangle_{Z_{1,R}} &= \langle c(\frac{\partial}{\partial u})\mu, \mu \rangle_{\partial Z_{1,R}} \\ &= -2i \|\phi' - C_1(\lambda)\phi\|_Y^2 + \langle c(\frac{\partial}{\partial u})\mu^{\text{nz}}, \mu^{\text{nz}} \rangle_{\partial Z_{1,R}}. \end{aligned}$$

By (3.106) and (3.108)-(3.110), we get

$$(3.111) \quad \begin{aligned} \|C_1(\lambda)\omega^{\text{zm},-}|_{\partial Z_{1,0}} - \omega^{\text{zm},+}|_{\partial Z_{1,0}}\|_Y^2 &= -i \langle c(\frac{\partial}{\partial u})\mu^{\text{nz}}, \mu^{\text{nz}} \rangle_{\partial Z_{1,R}} \\ &\leq \|\mu^{\text{nz}}\|_{\partial Z_{1,R}}^2 \leq \|\omega^{\text{nz}}\|_{\partial Z_{1,R}}^2 + \|\tilde{\omega}^{\text{nz}}\|_{\partial Z_{1,R}}^2. \end{aligned}$$

By (2.18), we have

$$(3.112) \quad \|\omega^{\text{nz}}\|_{\partial Z_{1,R}}^2 \leq e^{-\varepsilon R}\|\omega\|_{\partial Z_{1,0} \cup \partial Z_{2,0}}^2, \quad \|\tilde{\omega}^{\text{nz}}\|_{\partial Z_{1,R}}^2 \leq e^{-\varepsilon R}\|\tilde{\omega}\|_{\partial Z_{1,0}}^2.$$

By (3.56), we can show that there exists $C_1 > 0$, determined by Z_1, Z_2, F , such that

$$(3.113) \quad \|\omega\|_{\partial Z_{1,0} \cup \partial Z_{2,0}}^2 \leq C_1 \|\omega\|_{Z_{1,0} \cup Z_{2,0}}^2, \quad \|\tilde{\omega}\|_{\partial Z_{1,0}}^2 \leq C_1 \|\tilde{\omega}\|_{Z_{1,0}}^2.$$

By (2.33) and (3.106), we have

$$(3.114) \quad \|\tilde{\omega}\|_{Z_{1,0}}^2 \leq C_2 \|\phi - ic(\frac{\partial}{\partial u})\phi\|_Y^2 \leq C_2 \|\omega\|_{Z_{1,0}}^2.$$

Combining (3.111)-(3.114), we get (3.104) with $j = 1$. \square

Lemma 3.16. *For any $\varepsilon > 0$, there exist $R_0 > 0$, $c > 0$, such that, for any $R > R_0$, $\omega \in \Omega^\bullet(Z_R, F)$ an eigensection with eigenvalue $\lambda_0 \in]-\delta + \varepsilon, 0[\cup]0, \delta - \varepsilon[$, there exists $\tilde{\omega} \in \text{Im}(\mathcal{J}_{\lambda_0 - e^{-cR}, \lambda_0 + e^{-cR}}[,] - \delta, 0[\cup]0, \delta[, Z_R))$ satisfying*

$$(3.115) \quad \|\omega^{\text{zm}} - \tilde{\omega}^{\text{zm}}\|_{I_R Y} \leq e^{-cR} \|\omega\|_{Z_{1,0} \cup Z_{2,0}}.$$

Proof. We claim that there exist $c > 0$, $C > 0$, $R_0 > 0$, such that for any $R > R_0$, $\omega \in \Omega^\bullet(Z_R, F)$ an eigensection with eigenvalue $\lambda_0 \in]-\delta + \varepsilon, 0[\cup]0, \delta - \varepsilon[$, there exists $\mu \in \text{Im}(J_{\lambda_0 - e^{-cR}, \lambda_0 + e^{-cR}}[, Z_R))$, such that

$$(3.116) \quad \|\omega^{\text{zm}} - \mu^{\text{zm}}\|_{I_R Y} \leq e^{-cR} \|\omega\|_{Z_{1,0} \cup Z_{2,0}}, \quad \|\mu\|_{Z_{1,0} \cup Z_{2,0}} \leq C \|\omega\|_{Z_{1,0} \cup Z_{2,0}}.$$

Once (3.116) is proved, (3.115) follows : we may enlarge R_0 if necessary, then, by Theorem 3.10, we have

$$(3.117) \quad]\lambda_0 - e^{-cR}, \lambda_0 + e^{-cR}[\subseteq_{e^{-cR}}]-\delta, 0[\cup]0, \delta[.$$

Let $(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}}) \in \mathcal{E}_{\lambda_0 - e^{-cR}, \lambda_0 + e^{-cR}}[, R](Z_{12, \infty}, F)$, such that

$$(3.118) \quad \mu = J_{\lambda_0 - e^{-cR}, \lambda_0 + e^{-cR}}[, Z_R)(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}}).$$

By Definition 3.11, we have

$$(3.119) \quad \|\mu\|_{Z_{1,0} \cup Z_{2,0}} = \|(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}})\|_{\mathcal{E}_{\lambda_0 - e^{-cR}, \lambda_0 + e^{-cR}}[, R](Z_{12, \infty}, F)}.$$

Set

$$(3.120) \quad \tilde{\omega} = \mathcal{J}_{\lambda_0 - e^{-cR}, \lambda_0 + e^{-cR}}[,] - \delta, 0[\cup]0, \delta[, Z_R)(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}}).$$

by Proposition 3.13, (3.116), (3.117) and (3.119), we get (3.115).

Now we prove (3.116).

Since ω is an eigensection of $D_{Z_R}^F$ with eigenvalue λ_0 , we have

$$(3.121) \quad \omega^{\text{zm}} = e^{-i\lambda_0 u_1}(\omega^{\text{zm}, -}|_{\partial Z_{1,0}}) + e^{i\lambda_0 u_1}(\omega^{\text{zm}, +}|_{\partial Z_{1,0}}).$$

By Lemma 3.15, we have

$$(3.122) \quad \begin{aligned} \|C_1(\lambda_0)\omega^{\text{zm}, -}|_{\partial Z_{1,0}} - \omega^{\text{zm}, +}|_{\partial Z_{1,0}}\|_Y &\leq e^{-cR} \|\omega\|_{Z_{1,0} \cup Z_{2,0}}, \\ \|e^{4i\lambda_0 R} C_{12}(\lambda_0)\omega^{\text{zm}, -}|_{\partial Z_{1,0}} - \omega^{\text{zm}, -}|_{\partial Z_{1,0}}\|_Y &\leq e^{-cR} \|\omega\|_{Z_{1,0} \cup Z_{2,0}}. \end{aligned}$$

Same as (3.56), as a consequence of Trace Theorem and elliptic estimation, we have

$$(3.123) \quad \|\omega^{\text{zm}, -}|_{\partial Z_{1,0}}\|_Y \leq \|\omega|_{\partial Z_{1,0}}\|_Y \leq C \|\omega\|_{Z_{1,0} \cup Z_{2,0}}.$$

By (3.96) and (3.122), we have

$$(3.124) \quad \|\omega\|_{Z_{1,0} \cup Z_{2,0}} \leq C \|\omega^{\text{zm}, -}|_{\partial Z_{1,0}}\|_Y.$$

By Proposition 8.2, (3.122), (3.123) and (3.124), there exist $\phi_j \in \mathcal{H}^\bullet(Y, F[du])$, $\lambda_j \in \mathbb{R}$ and $\varphi_j \in \mathcal{H}^\bullet(Y, F[du])$ ($j = 1, \dots, \dim \mathcal{H}^\bullet(Y, F[du])$), such that, we have the following orthogonal decomposition

$$(3.125) \quad \omega^{\text{zm}, -}|_{\partial Z_{1,0}} = \sum_{j=1}^{\dim \mathcal{H}^\bullet(Y, F[du])} \phi_j ,$$

and

$$(3.126) \quad |\lambda_j - \lambda_0| < e^{-cR} , \quad \|\varphi_j - \phi_j\|_Y < e^{-cR} \|\omega\|_{Z_{1,0} \cup Z_{2,0}} , \\ e^{4iR\lambda_j} C_{12}(\lambda_j) \varphi_j = \varphi_j .$$

By (3.17) and (3.21), we can find $(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}}) \in \mathcal{E}_{A,R}(Z_{12,\infty}, F)$ satisfying

$$(3.127) \quad \omega_1^{\text{zm}} = \sum_{j=1}^{\dim \mathcal{H}^\bullet(Y, F[du])} (e^{-i\lambda_j u_1} \varphi_j + e^{i\lambda_j u_1} C_1(\lambda_j) \varphi_j) .$$

We take

$$(3.128) \quad \mu = J_{A,Z_R}(\omega_1, \omega_1^{\text{zm}}, \omega_2, \omega_2^{\text{zm}}) .$$

Then, under the natural identification $I_R Y \simeq I_{1,2R} Y \subseteq I_{1,\infty} Y$, we have

$$(3.129) \quad \mu^{\text{zm}} = \omega_1^{\text{zm}} .$$

For the first inequality in (3.116), by (3.121), (3.127) and (3.129), it suffices to show that, for $u_1 \in [0, 2R]$, we have

$$(3.130) \quad \left\| e^{-i\lambda_0 u_1} \left(\omega^{\text{zm}, -}|_{\partial Z_{1,0}} \right) + e^{i\lambda_0 u_1} \left(\omega^{\text{zm}, +}|_{\partial Z_{1,0}} \right) \right. \\ \left. - \sum_{j=1}^{\dim \mathcal{H}^\bullet(Y, F[du])} (e^{-i\lambda_j u_1} \varphi_j + e^{i\lambda_j u_1} C_1(\lambda_j) \varphi_j) \right\|_Y \leq e^{-cR} \|\omega\|_{Z_{1,0} \cup Z_{2,0}} .$$

This is a consequence of (3.122), (3.125) and (3.126).

For the second inequality in (3.116), by Definition 3.11, (2.33), (3.127) and (3.128), it suffices to show that

$$(3.131) \quad \sum_{j=1}^{\dim \mathcal{H}^\bullet(Y, F[du])} \|\varphi_j\|_Y \leq C \|\omega\|_{Z_{1,0} \cup Z_{2,0}} .$$

This follows from (3.123), (3.125) and (3.126). \square

Proposition 3.17. *For any $\varepsilon > 0$, there exist $R_0 > 0$, $c > 0$, such that for $R > R_0$, $B \subseteq_{e^{-cR}} A \subseteq]-\delta + \varepsilon, 0[\cup]0, \delta - \varepsilon[$, \mathcal{J}_{A,B,Z_R} is surjective.*

Proof. Suppose the contrary, i.e., there exist $R_i \rightarrow +\infty$, $\omega_i \in \Omega^\bullet(Z_{R_i}, F)$ and $\lambda_i \in B$ satisfying

$$(3.132) \quad D_{Z_{R_i}}^F \omega_i = \lambda_i \omega_i , \quad \omega_i \perp \text{Im}(\mathcal{J}_{A,B,Z_{R_i}}) .$$

By the construction of $\mathcal{J}_{A,B,Z_{R_i}}$ at the beginning of this subsection, we have

$$(3.133) \quad \text{Im} \left(\mathcal{J}_{A,]-\delta, 0[\cup]0, \delta[, Z_{R_i}} \right) = \text{Im} \left(\mathcal{J}_{A,B,Z_{R_i}} \right) \oplus \text{Im} \left(\mathcal{J}_{A,]-\delta, 0[\cup]0, \delta[\setminus B, Z_{R_i}} \right) .$$

Furthermore, $\mathcal{J}_{A,]-\delta, 0[\cup]0, \delta[\setminus B, Z_{R_i}}$ is spanned by some eigensections with eigenvalues in $] -\delta, 0[\cup]0, \delta[\setminus B$. Thus, by (3.132), we have

$$(3.134) \quad \omega_i \perp \text{Im}(\mathcal{J}_{A,]-\delta, 0[\cup]0, \delta[\setminus B, Z_{R_i}}) .$$

By multiplying suitable constants, we may assume that

$$(3.135) \quad \|\omega_i\|_{Z_{1,0} \cup Z_{2,0}} = 1 .$$

Then, by Proposition 3.14, we have

$$(3.136) \quad \|\omega_i^{\text{zm},+}\|_Y^2 + \|\omega_i^{\text{zm},-}\|_Y^2 \geq c > 0 .$$

By Lemma 3.16, there exists $\tilde{\omega}_i \in \text{Im}(\mathcal{J}_{A,]-\delta, 0[\cup]0, \delta[\setminus B, Z_{R_i}})$ such that

$$(3.137) \quad \|\omega_i^{\text{zm}} - \tilde{\omega}_i^{\text{zm}}\|_{I_{R_i}Y} \rightarrow 0 , \text{ as } i \rightarrow \infty .$$

By (3.136), (3.137), we have

$$(3.138) \quad \langle \omega_i^{\text{zm}}, \tilde{\omega}_i^{\text{zm}} \rangle \rightarrow \infty , \text{ as } i \rightarrow \infty .$$

By Lemma 2.1 and (3.135), there exists $C > 0$, such that

$$(3.139) \quad |\langle \omega_i, \tilde{\omega}_i \rangle - \langle \omega_i^{\text{zm}}, \tilde{\omega}_i^{\text{zm}} \rangle| \leq C ,$$

then, by (3.138), $\langle \omega_i, \tilde{\omega}_i \rangle$ tends to ∞ . Contradiction with (3.132). \square

Theorem 3.18. *For any $\varepsilon > 0$, there exists $R_0 > 0$, such that, for $R > R_0$, we have*

$$(3.140) \quad \text{Sp}(D_{Z_R}^F) \subseteq]-\infty, -R^{-1-\varepsilon}[\cup \{0\} \cup]R^{-1-\varepsilon}, \infty[.$$

Furthermore, if we denote

$$(3.141) \quad \begin{aligned} \Lambda_R \setminus \{0\} &= \left\{ \lambda_k : k \in \mathbb{Z} \setminus \{0\} \right\} , \quad \text{with } \dots \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \dots , \\ \text{Sp}(D_{Z_R}^F) \setminus \{0\} &= \left\{ \rho_k : k \in \mathbb{Z} \setminus \{0\} \right\} , \quad \text{with } \dots \leq \rho_{-1} < 0 < \rho_1 \leq \rho_2 \leq \dots , \end{aligned}$$

then, there exist $\gamma, c > 0$, such that, for $R > R_0$ and $|\lambda_k| < \gamma$, we have

$$(3.142) \quad |\lambda_k - \rho_k| \leq e^{-cR} .$$

Proof of Theorem 3.18. The proof of (3.140) is done in Theorem 3.10. We only prove the second part of the theorem.

We fix ε, c and R_0 such that Theorem 3.10, Proposition 3.13 and Proposition 3.17 hold. We enlarge R_0 such that, for $R > R_0$, we have

$$(3.143) \quad \varepsilon > R^{-1-\varepsilon} > e^{-cR}$$

By Theorem 8.1, we have

$$(3.144) \quad \Lambda_R = \bigcup_{k=1}^m \left\{ \lambda \in \mathbb{R} : 4R\lambda + \theta_k(\lambda) \in 2\pi\mathbb{Z} \right\} ,$$

where $\theta_1(\lambda), \dots, \theta_m(\lambda)$ are analytic functions on λ such that $e^{i\theta_1(\lambda)}, \dots, e^{i\theta_m(\lambda)}$ are all the eigenvalues of $C_{12}(\lambda)$. By enlarging R_0 , we can show that for $R > R_0$,

$$(3.145) \quad \Lambda_R \subseteq]-\infty, -R^{-1-\varepsilon}[\cup \{0\} \cup]R^{-1-\varepsilon}, \infty[.$$

For $k > 0$, if $\lambda_k < \delta - \varepsilon$, we apply Proposition 3.17 with

$$(3.146) \quad A =]0, \lambda_k[, \quad B =]R^{-1-\varepsilon}, \lambda_k - e^{-cR}[.$$

(By (3.143) and (3.145), we have $B \subseteq_{e^{-cR}} A$.) Then \mathcal{J}_{A,B,Z_R} is surjective. As consequence, $D_{Z_R}^F$ has at most $k - 1$ eigenvalues in B . Further, by Theorem 3.10, we have $\rho_1 > R^{-1-\varepsilon}$. Thus, we must have $\rho_k \geq \lambda_k - e^{-cR}$. A similar argument using Proposition 3.13 shows that $\rho_k \leq \lambda_k + e^{-cR}$. For $k < 0$, we have parallel arguments.

Set $\gamma = \delta - \varepsilon$, then (3.142) holds. \square

For $0 \leq p \leq \dim Z$, we set

$$(3.147) \quad \begin{aligned} C_{12}^p(\lambda) &= C_{12}(\lambda) \big|_{\mathcal{H}^p(Y,F) \oplus \mathcal{H}^{p-1}(Y,F) du}, \\ \Lambda_R^p &= \left\{ \lambda > 0 : \det(e^{4i\lambda R} C_{12}^p(\lambda) - 1) = 0 \right\}. \end{aligned}$$

Let $D_{Z_R}^{F,2,(p)}$ be the restriction of $D_{Z_R}^{F,2}$ on $\Omega^p(Z_R, F)$.

Theorem 3.19. *If we denote*

$$(3.148) \quad \begin{aligned} \Lambda_R^p &= \left\{ \lambda_k : k = 1, 2, \dots \right\}, \quad \text{with } 0 < \lambda_1 \leq \lambda_2 \leq \dots, \\ \text{Sp} \left(D_{Z_R}^{F,2,(p)} \right) \setminus \{0\} &= \left\{ \rho_k : k = 1, 2, \dots \right\}, \quad \text{with } 0 < \rho_1 \leq \rho_2 \leq \dots, \end{aligned}$$

then, there exist $\gamma, c > 0$, such that, for $R > R_0$ and $\lambda_k < \gamma$, we have

$$(3.149) \quad |\lambda_k^2 - \rho_k| \leq e^{-cR}.$$

Proof. If $A, B \subseteq \mathbb{R}$ are symmetric, i.e., $\lambda \in A$ implies $-\lambda \in A$ and same for B , then $\mathcal{E}_{A,R}(Z_{12,\infty}, F)$ and $\mathcal{E}_B(D_{Z_R}^F)$ are homogeneous. Let $\mathcal{E}_{A,R}^p(Z_{12,\infty}, F)$ and $\mathcal{E}_B^p(D_{Z_R}^F)$ be their degree p components. Furthermore, \mathcal{J}_{A,B,Z_R} preserves degree. Let $\mathcal{J}_{A,B,Z_R}^{(p)}$ be the restriction of \mathcal{J}_{A,B,Z_R} to $\mathcal{E}_{A,R}^p(Z_{12,\infty}, F)$. Then Proposition 3.13 and Proposition 3.17 hold for $\mathcal{J}_{A,B,Z_R}^{(p)}$. Noticing the fact that

$$(3.150) \quad \left\{ \lambda > 0 : \mathcal{E}_{\{\lambda, -\lambda\}, R}^p(Z_{12,\infty}, F) \neq 0 \right\} = \Lambda_R^p$$

the rest of the proof follows the same procedure as Theorem 3.18. \square

4. ASYMPTOTIC PROPERTIES OF THE SPECTRUM : BOUNDARY CASE

We use the notations in §3.1. We recall that the Riemannian manifolds $Z_{j,R} = Z_j \cup_Y [0, R] \times Y$ ($j = 1, 2$, $0 \leq R < \infty$) are defined in §0.2, and F is a flat vector bundle on $Z_{j,R}$. As stated in §0.2, we use the relative boundary condition on $\partial Z_{1,R}$ and the absolute boundary condition on $\partial Z_{2,R}$, which are defined by (1.5). We recall that $D_{Z_{j,R}}^F$ ($j = 1, 2$) are Hodge-de Rham operators acting $\Omega_{\text{bd}}^\bullet(Z_{j,R}, F)$. Let $\text{Sp} \left(D_{Z_{j,R}}^F \right)$ be the spectrum of $D_{Z_{j,R}}^F$. In this section, we give parallel results as §3 for $\text{Sp} \left(D_{Z_{j,R}}^F \right)$.

In §4.1, we establish results parallel to §3.3 and §3.4.

4.1. Approximating the kernel and small eigenvalues. We recall that $\mathcal{H}^\bullet(Z_{j,\infty}, F)$ and $\mathcal{H}_{\text{abs/rel}}^\bullet(Z_{j,\infty}, F) \subseteq \mathcal{H}^\bullet(Z_{j,\infty}, F)$ ($j = 1, 2$) are defined by (2.35) and (2.48). We use the convention $\mathcal{H}_{\text{bd}}^\bullet(Z_{1,\infty}, F) = \mathcal{H}_{\text{rel}}^\bullet(Z_{1,\infty}, F)$ and $\mathcal{H}_{\text{bd}}^\bullet(Z_{2,\infty}, F) = \mathcal{H}_{\text{abs}}^\bullet(Z_{2,\infty}, F)$.

We recall that $I_{j,R}Y \subseteq Z_{j,R}$ ($j = 1, 2$) are the cylindrical parts of $Z_{j,R}$, defined in §3.1. We recall that the following maps are defined in Definition 2.10,

$$(4.1) \quad \mathcal{R}_{d^F} : \mathcal{H}^\bullet(Z_{j,\infty}, F) \rightarrow \Omega^\bullet(I_{j,\infty}Y, F), \quad \text{for } j = 1, 2.$$

The inclusion $I_{j,R}Y \subseteq I_{j,\infty}Y$ induces

$$(4.2) \quad \Omega^\bullet(I_{j,\infty}Y, F) \rightarrow \Omega^\bullet(I_{j,R}Y, F).$$

Composing (4.1) and (4.2), we get

$$(4.3) \quad \mathcal{R}_{d^F,j} : \mathcal{H}^\bullet(Z_{j,\infty}, F) \rightarrow \Omega^\bullet(I_{j,R}Y, F), \quad \text{for } j = 1, 2.$$

We recall that $\chi_{j,R}$ ($j = 1, 2$) are defined by (3.24), which are smooth functions on I_RY . By restricting to $I_{j,R}Y \subseteq I_RY$ ($j = 1, 2$), we may view $\chi_{j,R}$ as smooth functions on $I_{j,R}Y$.

Parallel to Definition 3.1, we have the following definition.

Definition 4.1. We define

$$(4.4) \quad F_{Z_{j,R}} : \mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F) \rightarrow \Omega_{\text{bd}}^\bullet(Z_{j,R}, F),$$

by, for $(\omega, \hat{\omega}) \in \mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F)$,

$$(4.5) \quad F_{Z_{j,R}}(\omega, \hat{\omega})|_{Z_{j,0}} = \omega, \quad F_{Z_{j,R}}(\omega, \hat{\omega})|_{I_{j,R}Y} = d^F(\chi_{j,R} \mathcal{R}_{d^F,j}(\omega, \hat{\omega})) + \pi_Y^* \hat{\omega}.$$

By (2.42), $F_{Z_{j,R}}$ is well-defined. Furthermore, we have

$$(4.6) \quad d^F F_{Z_{j,R}}(\omega, \hat{\omega}) = 0.$$

We recall that $\varphi_R : Z_R \rightarrow Z$ is defined by (3.32). By restriction, we get

$$(4.7) \quad \varphi_{j,R} = \varphi_R|_{Z_{j,R}} : Z_{j,R} \rightarrow Z_j.$$

Then $\varphi_{j,R}$ ($j = 1, 2$) induce the canonical isomorphisms $H_{\text{bd}}^\bullet(Z_{j,R}, Y) \simeq H_{\text{bd}}^\bullet(Z_j, F)$.

Proposition 4.2. For $R > R' > 0$ and $\omega_1 \in \mathcal{H}_{L^2}^\bullet(Z_{1,\infty}, F)$, we have

$$(4.8) \quad [F_{Z_{1,R}}(\omega_1, 0)] = [F_{Z_{1,R'}}(\omega_1, 0)] \in H_{\text{bd}}^\bullet(Z_1, F).$$

For $R > R' > 0$ and $(\omega_2, \hat{\omega}) \in \mathcal{H}_{\text{bd}}^\bullet(Z_{2,\infty}, F)$, we have

$$(4.9) \quad [F_{Z_{2,R}}(\omega_2, \hat{\omega})] = [F_{Z_{2,R'}}(\omega_2, \hat{\omega})] \in H_{\text{bd}}^\bullet(Z_2, F).$$

We will prove Proposition 4.2 as a consequence of Proposition 3.3. For this, we need the following constructions.

Let $\overline{Z}_{j,R}$ ($j = 1, 2$) be another copy of $Z_{j,R}$. Set $Z_{j,R}^{\text{db}} = Z_{j,R} \cup_Y \overline{Z}_{j,R}$, which is a closed manifold. Then, $Z_{j,R}^{\text{db}}$ is equipped with a natural \mathbb{Z}_2 -action exchanging $Z_{j,R}$ and $\overline{Z}_{j,R}$. The flat vector bundle F on $Z_{j,R}$ and its copy on $\overline{Z}_{j,R}$ glue together giving a flat vector bundle on $Z_{j,R}^{\text{db}}$, which is still denoted by F . The \mathbb{Z}_2 -action lifts to F in the natural way. Let ι be the generator of this \mathbb{Z}_2 -action. Then, h^F and $\iota_* h^F$ glue together giving a Hermitian metric on F over $Z_{j,R}^{\text{db}}$, which is still denoted by h^F . Let $D_{Z_{j,R}^{\text{db}}}^F$ be the Hodge-de Rham operator acting on $\Omega^\bullet(Z_{j,R}^{\text{db}}, F)$. Then $D_{Z_{j,R}^{\text{db}}}^F$ is \mathbb{Z}_2 -equivariant.

Let ι^* be the action on $\Omega^\bullet(Z_{j,R}^{\text{db}}, F)$ or $H^\bullet(Z_{j,R}^{\text{db}}, F)$ induced by ι . Let $(\Omega^\bullet(Z_{j,R}^{\text{db}}, F))^\pm$ and $(H^\bullet(Z_{j,R}^{\text{db}}, F))^\pm$ be its eigenspaces with eigenvalue ± 1 . Then the injection $Z_{j,R}^{\text{db}} \hookrightarrow Z_{j,R}^{\text{db}}$ induces the following isomorphism

$$(4.10) \quad (\Omega^\bullet(Z_{j,R}^{\text{db}}, F))^{(-1)^j} \rightarrow \Omega_{\text{bd}}^\bullet(Z_{j,R}, F) .$$

Passing to cohomology, we get the isomorphism

$$(4.11) \quad (H^\bullet(Z_{j,R}^{\text{db}}, F))^{(-1)^j} \rightarrow H_{\text{bd}}^\bullet(Z_{j,R}, F) .$$

Proof of Proposition 4.2. Let $\mathcal{H}^\bullet(Z_{j,\infty}^{\text{db}}, F)$ be $\mathcal{H}^\bullet(Z_{12,\infty}, F)$ defined in §3.2 with $Z_{1,\infty}$ and $Z_{2,\infty}$ replaced by $Z_{j,\infty}$ and $\overline{Z}_{j,\infty}$. More precisely,

$$(4.12) \quad \mathcal{H}^\bullet(Z_{j,\infty}^{\text{db}}, F) = \left\{ (\omega_1, \omega_2, \hat{\omega}) : (\omega_1, \hat{\omega}) \in \mathcal{H}^\bullet(Z_{j,\infty}, F), (\omega_2, \hat{\omega}) \in \mathcal{H}^\bullet(\overline{Z}_{j,\infty}, F) \right\} ,$$

By Definition 3.1, we have the following map

$$(4.13) \quad F_{Z_{j,R}^{\text{db}}} : \mathcal{H}^\bullet(Z_{j,\infty}^{\text{db}}, F) \rightarrow \Omega^\bullet(Z_{j,R}^{\text{db}}, F) .$$

Let N^{du} be the number operator on $\mathcal{H}^\bullet(Y, F[du])$ associated to the variable du , i.e., its restriction to $\mathcal{H}^\bullet(Y, F)$ is zero, its restriction to $\mathcal{H}^\bullet(Y, F)du$ is identity. We define the following involution

$$(4.14) \quad \begin{aligned} \iota^{\mathcal{H}} : \mathcal{H}^\bullet(Z_{j,\infty}^{\text{db}}, F) &\rightarrow \mathcal{H}^\bullet(Z_{j,\infty}^{\text{db}}, F) \\ (\omega_1, \omega_2, \hat{\omega}) &\mapsto (\omega_2, \omega_1, (-1)^{N^{du}} \hat{\omega}) . \end{aligned}$$

The following diagram commutes

$$(4.15) \quad \begin{array}{ccc} \mathcal{H}^\bullet(Z_{j,\infty}^{\text{db}}, F) & \xrightarrow{\iota^{\mathcal{H}}} & \mathcal{H}^\bullet(Z_{j,\infty}^{\text{db}}, F) \\ \downarrow F_{Z_{j,R}^{\text{db}}} & & \downarrow F_{Z_{j,R}^{\text{db}}} \\ \Omega^\bullet(Z_{j,R}^{\text{db}}, F) & \xrightarrow{\iota^*} & \Omega^\bullet(Z_{j,R}^{\text{db}}, F) \end{array} .$$

Let $(\mathcal{H}^\bullet(Z_{j,\infty}^{\text{db}}, F))^\pm$ be the eigenspace of $\iota^{\mathcal{H}}$ with eigenvalue ± 1 , then, we get

$$(4.16) \quad F_{Z_{j,R}^{\text{db}}} : (\mathcal{H}^\bullet(Z_{j,\infty}^{\text{db}}, F))^\pm \rightarrow (\Omega^\bullet(Z_{j,R}^{\text{db}}, F))^\pm .$$

We have also the following isomorphisms

$$(4.17) \quad \begin{aligned} \mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F) &\rightarrow (\mathcal{H}^\bullet(Z_{j,\infty}^{\text{db}}, F))^{(-1)^j} \\ (\omega, \hat{\omega}) &\mapsto (\omega, (-1)^j \omega, \hat{\omega}) . \end{aligned}$$

The following diagram commutes

$$(4.18) \quad \begin{array}{ccc} (\mathcal{H}^\bullet(Z_{j,\infty}^{\text{db}}, F))^{(-1)^j} & \xrightarrow{F_{Z_{j,R}^{\text{db}}}} & (\Omega^\bullet(Z_{j,R}^{\text{db}}, F))^{(-1)^j} \\ \downarrow & & \downarrow \\ \mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F) & \xrightarrow{F_{Z_{j,R}}} & \Omega_{\text{bd}}^\bullet(Z_{j,R}, F) \end{array}$$

where the left vertical map is defined by (4.17) and the right vertical map is induced by the injection $Z_{j,R} \hookrightarrow Z_{j,R}^{\text{db}}$.

By (4.11) and (4.18), the present proposition follows from Proposition 3.3 with Z_R replaced by $Z_{j,R}^{\text{db}}$. \square

In the rest of this section, we state several results parallel to those in §3.3 and §3.4. Their proofs follow the strategy as the proof of Proposition 4.2 : on $Z_{j,R}^{\text{db}}$, the constructions in §3 commute with the action of ι , and the objects concerned associated with $Z_{j,R}$ (eigenspace of Hodge-de Rham operator, cohomology, etc.) are canonically isomorphic to the eigenspaces of ι with eigenvalue $(-1)^j$ in the corresponding objects associated with $Z_{j,R}^{\text{db}}$.

Recall that the L^2 -norm $\|\cdot\|$ is defined in §0.4. For any $(\omega, \hat{\omega}) \in \mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F)$, set

$$(4.19) \quad \|(\omega, \hat{\omega})\|_{\mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F), R}^2 = \|\omega\|_{Z_{j,R}}^2.$$

By passing to $Z_{j,R}^{\text{db}}$ and applying Proposition 3.4, we get the following proposition.

Proposition 4.3. *There exist $c > 0$, $R_0 > 0$, such that, for any $R > R_0$, $(\omega, \hat{\omega}) \in \mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F)$ ($j = 1, 2$), we have*

$$(4.20) \quad 1 - e^{-cR} \leq \frac{\|F_{Z_{j,R}}(\omega, \hat{\omega})\|_{Z_{j,R}}}{\|(\omega, \hat{\omega})\|_{\mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F), R}} \leq 1 + e^{-cR}.$$

Let

$$(4.21) \quad P^{\ker(D_{Z_{j,R}}^{F,2})} : \Omega_{\text{bd}}^\bullet(Z_{j,R}, F) \rightarrow \ker(D_{Z_{j,R}}^{F,2})$$

be the orthogonal projections.

Definition 4.4. For $j = 1, 2$, set

$$(4.22) \quad \mathcal{F}_{Z_{j,R}} = P^{\ker(D_{Z_{j,R}}^{F,2})} \circ F_{Z_{j,R}} : \mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F) \rightarrow \ker(D_{Z_{j,R}}^{F,2}).$$

By passing to $Z_{j,R}^{\text{db}}$ and applying Proposition 3.7, we get the following proposition.

Proposition 4.5. *There exist $c > 0$, $R_0 > 0$, such that, for any $R > R_0$, $(\omega, \hat{\omega}) \in \mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F)$ ($j = 1, 2$), we have*

$$(4.23) \quad \|(F_{Z_{j,R}} - \mathcal{F}_{Z_{j,R}})(\omega, \hat{\omega})\|_{\mathcal{C}^0(Z_{j,R})} \leq e^{-cR} \|(\omega, \hat{\omega})\|_{\mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F)}.$$

By passing to $Z_{j,R}^{\text{db}}$ and applying Theorem 3.10, we get the following theorem.

Theorem 4.6. *There exists $R_0 > 0$ such that, for $R > R_0$, the maps $\mathcal{F}_{Z_{j,R}}$ ($j = 1, 2$) is bijective, and*

$$(4.24) \quad \text{Sp}(D_{Z_{j,R}}^F) \subseteq]-\infty, -R^{-1-\varepsilon}[\cup \{0\} \cup]R^{-1-\varepsilon}, +\infty[.$$

Set

$$(4.25) \quad C_{j,\text{bd}}(\lambda) = (-1)^j \left(C_j(\lambda)|_{\mathcal{H}^\bullet(Y,F)} - C_j(\lambda)|_{\mathcal{H}^\bullet(Y,F)_{du}} \right).$$

For any $R \geq 0$, set

$$(4.26) \quad \Lambda_{j,R} = \left\{ \lambda \in \mathbb{R}, \det \left(e^{2i\lambda R} C_{j,\text{bd}}(\lambda)|_{\mathcal{H}^\bullet(Y,F)} - 1 \right) = 0 \right\}, \quad \text{for } j = 1, 2.$$

By passing to $Z_{j,R}^{\text{db}}$ and applying Theorem 3.18, we get the following theorem.

Theorem 4.7. *Theorem 3.18 holds for $(\text{Sp}(D_{Z_{j,R}}^F), \Lambda_{j,R})$, where $j = 1, 2$.*

For $0 \leq p \leq \dim Z$, set

$$(4.27) \quad \begin{aligned} C_{j,\text{bd}}^p(\lambda) &= C_{j,\text{bd}}(\lambda) \Big|_{\mathcal{H}^p(Y,F) \oplus \mathcal{H}^{p-1}(Y,F) du}, \quad \text{for } j = 1, 2, \\ \Lambda_{j,R}^p &= \left\{ \lambda \in \mathbb{R}, \det \left(e^{2i\lambda R} C_{j,\text{bd}}^p(\lambda) \Big|_{\mathcal{H}^p(Y,F)} - 1 \right) = 0 \right\}. \end{aligned}$$

By passing to $Z_{j,R}^{\text{db}}$ and applying Theorem 3.19, we get the following theorem.

Theorem 4.8. *Theorem 3.19 holds for $(\text{Sp}(D_{Z_{j,R}}^{F,2,(p)}), \Lambda_{j,R}^p)$, where $j = 1, 2$.*

5. ASYMPTOTICS OF THE (WEIGHTED) ZETA DETERMINANTS

The purpose of this section is to prove Theorem 0.1.

In this section, we use notations in §3.1. For convenience, we use the following convention : $Z_{0,R} = Z_R$, $\zeta_{0,R} = \zeta_R$, and so forth, i.e., we add also a sub-index 0 to objects associated to Z_R . And we use the following definition of ζ -functions $\zeta_{j,R}(s)$ ($j = 0, 1, 2$), which is equivalent to (0.5).

$$(5.1) \quad \zeta_{j,R}(s) = -\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr} \left[(-1)^N N \exp \left(-t D_{Z_{j,R}}^{F,2} \right) \left(1 - P^{\ker(D_{Z_{j,R}}^{F,2}}) \right) \right] dt.$$

Let $\varepsilon \in]0, 1[$. Let $\zeta_{j,R}^S(s)$ (resp. $\zeta_{j,R}^L(s)$) be the contribution of $\int_0^{R^{2-\varepsilon}}$ (resp. $\int_{R^{2-\varepsilon}}^\infty$) to $\zeta_{j,R}(s)$ in (5.1). Then

$$(5.2) \quad \zeta_{j,R} = \zeta_{j,R}^S + \zeta_{j,R}^L.$$

In §5.1, we define model operators which will serve as the limit (as $R \rightarrow \infty$) of the Hodge-de Rham operators concerned. In §5.2, we treat the contributions of $\zeta_{j,R}^S$. In §5.3, we treat the contributions of $\zeta_{j,R}^L$.

5.1. Model operators. Let $I_{1,R} = [-R, 0]$, $I_{2,R} = [0, R]$ and $I_R = [-R, R]$. Let u be the coordinate. And we sometimes add a sub-index 0 to objects associated to $I_{0,R} := I_R$.

We recall that $\mathcal{H}^\bullet(Y, F)$ and $\mathcal{H}^\bullet(Y, F[du])$ are defined by (2.1) and (2.6). Let $\Omega^\bullet(I_R, \mathcal{H}^\bullet(Y, F))$ be the vector space of differential forms on I_R with values in $\mathcal{H}^\bullet(Y, F)$. We define the total degree of $\omega \in \Omega^p(I_R, \mathcal{H}^q(Y, F))$ to be $p + q$. We have the canonical identification

$$(5.3) \quad \Omega^\bullet(I_R, \mathcal{H}^\bullet(Y, F)) \simeq \mathcal{C}^\infty(I_R, \mathcal{H}^\bullet(Y, F[du])).$$

For $\omega \in \Omega^\bullet(I_R, \mathcal{H}^\bullet(Y, F))$, let $u \mapsto \omega_u \in \mathcal{H}^\bullet(Y, F[du])$ be the corresponding function.

We recall that the operator $c(\frac{\partial}{\partial u})$ acting on $\mathcal{H}^\bullet(Y, F[du])$ is defined by (2.4) and that $\mathcal{L}_j^\bullet \subseteq \mathcal{H}^\bullet(Y, F[du])$ ($j = 1, 2$) are defined at the beginning of §3.2. We define the model operator D_{I_R} by

$$(5.4) \quad D_{I_R} = c(\frac{\partial}{\partial u}) \frac{\partial}{\partial u},$$

with

$$(5.5) \quad \text{Dom}(D_{I_R}) = \left\{ \omega \in \Omega^\bullet(I_R, \mathcal{H}^\bullet(Y, F)) : \omega_{-R} \in \mathcal{L}_1^\bullet, \omega_R \in \mathcal{L}_2^\bullet \right\}.$$

We define equally $D_{I_{1,R}}$ and $D_{I_{2,R}}$ with

$$(5.6) \quad \begin{aligned} \text{Dom}(D_{I_{1,R}}) &= \left\{ \omega \in \Omega^\bullet(I_{1,R}, \mathcal{H}^\bullet(Y, F)) : \omega_{-R} \in \mathcal{L}_1^\bullet, \omega_0 \in \mathcal{H}^\bullet(Y, F)du \right\}, \\ \text{Dom}(D_{I_{2,R}}) &= \left\{ \omega \in \Omega^\bullet(I_{2,R}, \mathcal{H}^\bullet(Y, F)) : \omega_R \in \mathcal{L}_2^\bullet, \omega_0 \in \mathcal{H}^\bullet(Y, F) \right\}. \end{aligned}$$

We remark that $D_{I_{j,R}}^2$ ($j = 0, 1, 2$) preserve the total degree. Let $D_{I_{j,R}}^{2,(p)}$ be its restriction to total degree p .

Let $\mathcal{L}_{j,\text{abs}/\text{rel}}^\bullet$ be absolute/relative part of \mathcal{L}_j^\bullet , which is defined by (2.46). We use the convention $\mathcal{L}_{1,\text{bd}}^\bullet = \mathcal{L}_{1,\text{rel}}^\bullet$ and $\mathcal{L}_{2,\text{bd}}^\bullet = \mathcal{L}_{2,\text{abs}}^\bullet$. By (5.4), (5.5) and (5.6), we have

$$(5.7) \quad \ker(D_{I_R}^{2,(p)}) = \mathcal{L}_1^p \cap \mathcal{L}_2^p, \quad \ker(D_{I_{j,R}}^{2,(p)}) = \mathcal{L}_{j,\text{bd}}^p, \quad \text{for } j = 1, 2,$$

where the vectors in $\mathcal{L}_1^p \cap \mathcal{L}_2^p$ (resp. $\mathcal{L}_{j,\text{bd}}^p$) are viewed as constant functions on I_R (resp. $I_{j,R}$).

We define the composition map

$$(5.8) \quad \alpha_{p,\mathcal{L}} : \mathcal{L}_{1,\text{rel}}^p \rightarrow \mathcal{L}_{1,\text{rel}}^p \cap \mathcal{L}_{2,\text{rel}}^p \rightarrow \mathcal{L}_1^p \cap \mathcal{L}_2^p,$$

where the first map is orthogonal projection, and the second one is injection. We also define

$$(5.9) \quad \beta_{p,\mathcal{L}} : \mathcal{L}_1^p \cap \mathcal{L}_2^p \rightarrow \mathcal{L}_{1,\text{abs}}^p \cap \mathcal{L}_{2,\text{abs}}^p \rightarrow \mathcal{L}_{2,\text{abs}}^p,$$

which is still the composition of orthogonal projection and injection. And

$$(5.10) \quad \delta_{p,\mathcal{L}} : \mathcal{L}_{2,\text{abs}}^p \rightarrow \mathcal{L}_{2,\text{rel}}^{p+1,\perp} \rightarrow \mathcal{L}_{1,\text{rel}}^{p+1} \cap \mathcal{L}_{2,\text{rel}}^{p+1,\perp} \rightarrow \mathcal{L}_{1,\text{rel}}^{p+1},$$

where the first map is the $du \wedge$ operation (cf. (2.4)), the second one is orthogonal projection and the last one is injection. Then we get the following exact sequence

$$(5.11) \quad \cdots \longrightarrow \mathcal{L}_{1,\text{bd}}^p \xrightarrow{\alpha_{p,\mathcal{L}}} \mathcal{L}_1^p \cap \mathcal{L}_2^p \xrightarrow{\beta_{p,\mathcal{L}}} \mathcal{L}_{2,\text{bd}}^p \xrightarrow{\delta_{p,\mathcal{L}}} \cdots.$$

The exactness of (5.11) is justified by the following identities

$$(5.12) \quad \begin{aligned} \text{Im}(\alpha_{p,\mathcal{L}}) &= \ker(\beta_{p,\mathcal{L}}) = \mathcal{L}_{1,\text{rel}}^p \cap \mathcal{L}_{2,\text{rel}}^p, \\ \text{Im}(\beta_{p,\mathcal{L}}) &= \ker(\delta_{p,\mathcal{L}}) = \mathcal{L}_{1,\text{abs}}^p \cap \mathcal{L}_{2,\text{abs}}^p, \\ \text{Im}(\delta_{p,\mathcal{L}}) &= \ker(\alpha_{p+1,\mathcal{L}}) = \mathcal{L}_{1,\text{rel}}^{p+1} \cap \mathcal{L}_{2,\text{rel}}^{p+1,\perp}. \end{aligned}$$

We may view (5.11) as the Mayer-Vietoris sequence in this model setting.

Recall that $C_j(\lambda) \in \text{End}(\mathcal{H}^\bullet(Y, F[du]))$ ($j = 1, 2$) are the scattering matrices associated to $D_{Z_{j,\infty}}^F$ (cf. §3.2) and that the operators $C_{12}(\lambda)$ and $C_{j,\text{bd}}(\lambda)$ are introduced in (3.18) and (4.25). For any of these operators $C(\lambda)$, let $C = C(0)$ and C^p be its restriction to $\mathcal{H}^p(Y, F) \oplus \mathcal{H}^{p-1}(Y, F)du$.

By (2.45) and (2.46), we have

$$(5.13) \quad \begin{aligned} \ker(C_{1,\text{bd}}^p - 1) &= \mathcal{L}_{1,\text{rel}}^p \oplus i \frac{\partial}{\partial u} \mathcal{L}_{1,\text{rel}}^{p+1}, \\ \ker(C_{2,\text{bd}}^p - 1) &= \mathcal{L}_{2,\text{abs}}^p \oplus du \wedge \mathcal{L}_{2,\text{abs}}^{p-1}, \\ \ker(C_{12}^p - 1) &= (\mathcal{L}_1^p \cap \mathcal{L}_2^p) \oplus i \frac{\partial}{\partial u} (\mathcal{L}_{1,\text{rel}}^{p+1} \cap \mathcal{L}_{2,\text{rel}}^{p+1}) \oplus du \wedge (\mathcal{L}_{1,\text{abs}}^{p-1} \cap \mathcal{L}_{2,\text{abs}}^{p-1}). \end{aligned}$$

For $C = C_{12}$ or $C_{j,\mathbf{bd}}$ ($j = 1, 2$), set

$$(5.14) \quad \chi'(C) = \sum_p (-1)^p p \dim \ker (C^p - 1) .$$

We recall that χ' is defined in (0.8).

Lemma 5.1. *We have*

$$(5.15) \quad \chi'(C_{12}) - \chi'(C_{1,\mathbf{bd}}) - \chi'(C_{2,\mathbf{bd}}) = 2\chi'.$$

Proof. We denote

$$(5.16) \quad \begin{aligned} \dim \mathcal{L}_{1,\mathbf{abs}}^p &= x_p, & \dim \mathcal{L}_{2,\mathbf{abs}}^p &= y_p, \\ \dim(\mathcal{L}_{1,\mathbf{abs}}^p \cap \mathcal{L}_{2,\mathbf{abs}}^p) &= u_p, & \dim(\mathcal{L}_{1,\mathbf{abs}}^{p,\perp} \cap \mathcal{L}_{2,\mathbf{abs}}^{p,\perp}) &= v_p, \\ \dim \mathcal{H}^p(Y, F) &= h_p. \end{aligned}$$

Then, by (2.46), we have

$$(5.17) \quad \dim \mathcal{L}_{1,\mathbf{rel}}^{p+1} = h_p - x_p, \quad \dim \mathcal{L}_{2,\mathbf{rel}}^{p+1} = h_p - y_p, \quad \dim(\mathcal{L}_{1,\mathbf{rel}}^{p+1} \cap \mathcal{L}_{2,\mathbf{rel}}^{p+1}) = v_p .$$

Since $\mathcal{H}^p(Y, F) = (\mathcal{L}_{1,\mathbf{abs}}^p + \mathcal{L}_{2,\mathbf{abs}}^p) \oplus (\mathcal{L}_{1,\mathbf{abs}}^{p,\perp} \cap \mathcal{L}_{2,\mathbf{abs}}^{p,\perp})$, we get

$$(5.18) \quad h_p = x_p + y_p - u_p + v_p .$$

By (5.13), (5.14), (5.17) and (5.18), we have

$$(5.19) \quad \begin{aligned} \chi'(C_{12}) - \chi'(C_{1,\mathbf{bd}}) - \chi'(C_{2,\mathbf{bd}}) &= \sum_p 2(-1)^p (y_p - u_p) , \\ \dim \mathcal{L}_1^p \cap \mathcal{L}_2^p - \dim \mathcal{L}_{1,\mathbf{bd}}^p - \dim \mathcal{L}_{2,\mathbf{bd}}^p &= \sum_p (-1)^p (y_p - u_p) . \end{aligned}$$

By (0.8) and (5.19), it rests to show that

$$(5.20) \quad \begin{aligned} &\dim \mathcal{L}_1^p \cap \mathcal{L}_2^p - \dim \mathcal{L}_{1,\mathbf{bd}}^p - \dim \mathcal{L}_{2,\mathbf{bd}}^p \\ &= \dim H^p(Z, F) - \dim H_{\mathbf{bd}}^p(Z_1, F) - \dim H_{\mathbf{bd}}^p(Z_2, F) . \end{aligned}$$

By Theorem 1.1, Theorem 3.10 and Theorem 4.6, (5.20) is equivalent to

$$(5.21) \quad \begin{aligned} &\dim \mathcal{L}_1^p \cap \mathcal{L}_2^p - \dim \mathcal{L}_{1,\mathbf{bd}}^p - \dim \mathcal{L}_{2,\mathbf{bd}}^p \\ &= \dim \mathcal{H}^p(Z_{12,\infty}, F) - \dim \mathcal{H}_{\mathbf{bd}}^p(Z_{1,\infty}, F) - \dim \mathcal{H}_{\mathbf{bd}}^p(Z_{2,\infty}, F) . \end{aligned}$$

This follows from (2.49) and (3.13). □

We denote

$$(5.22) \quad a_p = \dim \operatorname{Im}(\alpha_{p,\mathcal{L}}), \quad b_p = \dim \operatorname{Im}(\beta_{p,\mathcal{L}}), \quad d_p = \dim \operatorname{Im}(\delta_{p,\mathcal{L}}) .$$

Lemma 5.2. *We have*

$$(5.23) \quad \chi' = \sum_p (-1)^p d_p, \quad \chi'(C_{12}) = \sum_p (-1)^p (a_p - b_p) .$$

Proof. Same as Lemma 5.1, all the terms involved can be expressed by x_p, y_p, u_p, v_p . Then (5.23) follows from a direct calculation. □

We turn to study the spectra and ζ -functions in this model setting.

For any $R \geq 0$, set

$$(5.24) \quad \begin{aligned} \Lambda_R^{*,p} &= \left\{ \lambda > 0 : \det(e^{4i\lambda R} C_{12}^p - 1) = 0 \right\}, \\ \Lambda_{j,R}^{*,p} &= \left\{ \lambda > 0 : \det(e^{2i\lambda R} C_{j,\mathbf{bd}}^p - 1) = 0 \right\}, \quad \text{for } j = 1, 2. \end{aligned}$$

Proposition 5.3. *We have*

$$(5.25) \quad \text{Sp} \left(D_{I_{j,R}}^{2,(p)} \right) \setminus \{0\} = \left\{ \lambda^2 : \lambda \in \Lambda_{j,R}^{*,p} \right\}, \quad \text{for } j = 0, 1, 2.$$

Proof. Firstly, we consider the case $j = 0$.

By shifting the coordinate, we identify $I_{1,R}$ to $[0, R]$. We define $I_{1,\infty} = [0, \infty[$. Let $D_{I_{1,\infty}}$ be the operator defined by (5.4) with the same boundary condition (only at $u = 0$) as $D_{I_{1,R}}$ for $R < \infty$. Here, $D_{I_{1,\infty}}$ is exactly the $D_{Z_\infty}^F$ constructed in §2.3 with Z_∞ replaced by $I_{1,\infty}$ and F replaced by $\mathcal{H}^\bullet(Y, F)$. Using (2.45) and (2.46), a direct calculation shows that a generalized eigensection of $D_{I_{1,\infty}}$ with eigenvalue $\lambda \neq 0$ takes the following form

$$(5.26) \quad e^{-i\lambda u} (1 - ic(\frac{\partial}{\partial u}))\phi + e^{i\lambda u} C_1 (1 - ic(\frac{\partial}{\partial u}))\phi, \quad \phi \in \mathcal{H}^\bullet(Y, F).$$

Comparing to (2.31), we see that there are only zero modes (cf. (2.14), (2.15)). Furthermore, the scattering matrix of $D_{I_{1,\infty}}$ is C_1 , which does not depend on λ .

We construct equally $D_{I_{2,\infty}}$. And its scattering matrix is C_2 .

With the above constructions, we are almost in a special case of the problem considered in §3. The only difference is that I_R is not a closed manifold. Checking all the arguments in §3, we see that they still work for D_{I_R} . Now, applying Theorem 3.19, we see that $\text{Sp} \left(D_{I_R}^{2,(p)} \right) \setminus \{0\}$ is approximated by $\Lambda_R^{*,p}$ in the sense of (3.149). Notice that the error terms in the whole argument leading to Theorem 3.19 come from non zero modes. Here, since there are only zero modes, the approximation is replaced by equality. This proves (5.25).

For $j = 1, 2$, replacing Theorem 3.19 by Theorem 4.7, the same argument works. \square

Let $\zeta_{*,j,R}(s)$ be the ζ -functions of $D_{I_{j,R}}^2$ defined in the same way as (5.1).

Proposition 5.4. *We have*

$$(5.27) \quad \begin{aligned} \zeta_{*,R}'(0) &= \chi'(C_{12}) \log(2R) - \chi(Y, F) \log 2 + \sum_{p=0}^{\dim Y} \frac{p}{2} (-1)^p \log \det^* \left(\frac{2 - C_{12}^p - (C_{12}^p)^{-1}}{4} \right), \\ \zeta_{*,j,R}'(0) &= \chi'(C_{j,\mathbf{bd}}) \log R - \chi(Y, F) \log 2, \quad \text{for } j = 1, 2. \end{aligned}$$

Proof. Applying (0.5) and (5.25), both identities are consequences of Appendix (8.16). The first identity is the weighted sum of (8.16) with V replaced by $\mathcal{H}^p(Y, F) \oplus \mathcal{H}^{p-1}(Y, F)du$ and C replaced by C_{12}^p . For the second identity, we replace C by $C_{j,\mathbf{bd}}^p$ and replace R by $R/2$. Since $\text{Sp} \left(C_{j,\mathbf{bd}}^p \right) \subseteq \{-1, 1\}$, the $\log \det^*$ term vanishes. \square

5.2. Small time contribution. We denote

$$(5.28) \quad \begin{aligned} \Theta_R(t) &= \sum_{j=0}^2 (-1)^{(j-1)(j-2)/2} \operatorname{Tr} \left[(-1)^N N \exp \left(-t D_{Z_{j,R}}^{F,2} \right) \right], \\ \Theta_R^*(t) &= \sum_{j=0}^2 (-1)^{(j-1)(j-2)/2} \operatorname{Tr} \left[(-1)^N N \exp \left(-t D_{I_{j,R}}^2 \right) \right]. \end{aligned}$$

Same as $\zeta_R^{S/L}$ defined at the beginning of the section, we define $\zeta_{*,j,R}^{S/L}$ ($j = 0, 1, 2$). By (5.1) and (5.15), we have

$$(5.29) \quad \sum_{j=0}^2 (-1)^{(j-1)(j-2)/2} \left(\zeta_{j,R}^S(s) - \zeta_{*,j,R}^S(s) \right) = -\frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} t^{s-1} (\Theta_R(t) - \Theta_R^*(t)) dt.$$

Theorem 5.5. *There exist $c > 0$ such that as $R \rightarrow \infty$,*

$$(5.30) \quad \sum_{j=0}^2 (-1)^{(j-1)(j-2)/2} \left(\zeta_{j,R}^{S'}(0) - \zeta_{*,j,R}^{S'}(0) \right) = \mathcal{O}(e^{-cR^{\varepsilon/2}}).$$

Proof. Let $f \in \mathcal{C}^\infty(\mathbb{R})$ be an even function such that $f(u) = 1$ for $|u| \leq 1/2$ and $f(u) = 0$ for $|u| \geq 1$. Same as [8, §13(b)], for $t, \varsigma > 0$ and $z \in \mathbb{C}$, set

$$(5.31) \quad \begin{aligned} F_{t,\varsigma}(z) &= \int_{-\infty}^{\infty} e^{i\sqrt{2}\varsigma v z} e^{-\frac{1}{2}v^2} f(\sqrt{\varsigma} t v) \frac{dv}{\sqrt{2\pi}}, \\ G_{t,\varsigma}(z) &= \int_{-\infty}^{\infty} e^{i\sqrt{2}\varsigma v z} e^{-\frac{1}{2}v^2/t} (1 - f(\sqrt{\varsigma} v)) \frac{dv}{\sqrt{2\pi t}}. \end{aligned}$$

Then

$$(5.32) \quad F_{t,\varsigma}(\sqrt{t}z) + G_{t,\varsigma}(z) = \exp(-tz^2).$$

Let

$$(5.33) \quad \begin{aligned} &F_{t,\varsigma} \left(\sqrt{t} D_{Z_{j,R}}^F \right) (x, y), \quad G_{t,\varsigma} \left(D_{Z_{j,R}}^F \right) (x, y) \\ &\in (\Lambda^\bullet(T^*Z_{j,R}) \otimes F)_x \otimes (\Lambda^\bullet(T^*Z_{j,R}) \otimes F)_y^* \end{aligned}$$

be the smooth kernel of operators $F_{t,\varsigma} \left(\sqrt{t} D_{Z_{j,R}}^F \right)$ and $G_{t,\varsigma} \left(D_{Z_{j,R}}^F \right)$ with respect to the volume form induced by the Riemannian metric on $Z_{j,R}$.

By the construction of $G_{t,\varsigma}(z)$, for any $k \in \mathbb{N}$, there exists $c, C > 0$, such that, for any $t > 0$ and $0 < \varsigma < 1$, we have (cf. [27, (1.6.16)])

$$(5.34) \quad \sup_{z \in \mathbb{C}} |z^k G_{t,\varsigma}(z)| \leq C e^{-c/\varsigma t}.$$

As a consequence, for any $k, k' \in \mathbb{N}$, there exists $c, C > 0$, such that, for $0 < t < R^{2-\varepsilon}$, $0 < \varsigma < R^{-2+\varepsilon/2}$ and $j = 0, 1, 2$, we have

$$(5.35) \quad \left\| D_{Z_{j,R}}^{F,k} G_{t,\varsigma} \left(D_{Z_{j,R}}^F \right) D_{Z_{j,R}}^{F,k'} \right\|_{0,0} \leq C t e^{-cR^{\varepsilon/2}},$$

where $\|\cdot\|_{0,0}$ is the operator norm induced by the L^2 -norm. Then, by Proposition 3.5 and (5.35), there exists $c, C > 0$, such that, for $0 < t < R^{2-\varepsilon}$, $0 < \varsigma < R^{-2+\varepsilon/2}$, $j = 0, 1, 2$ and $x, y \in Z_{j,R}$, we have

$$(5.36) \quad \left| G_{t,\varsigma} \left(D_{Z_{j,R}}^F \right) (x, y) \right| \leq C t e^{-c R^{\varepsilon/2}} .$$

By the finite propagation speed principal (cf. [37, §2.6, Theorem 6.1], [27, Appendix D.2]), if the distance between x and y is greater than $\varsigma^{-1/2}$, then $F_{t,\varsigma} \left(\sqrt{t} D_{Z_{j,R}}^F \right) (x, y) = 0$. In the rest of the proof, we take $\varsigma = R^{-2+\varepsilon/3}$ and suppose that R is large enough. For $x \in Z_{j,R/2} \subseteq Z_{j,R} \subseteq Z_R$ ($j = 1, 2$), we have

$$(5.37) \quad F_{t,\varsigma} \left(\sqrt{t} D_{Z_{j,R}}^F \right) (x, x) = F_{t,\varsigma} \left(\sqrt{t} D_{Z_R}^F \right) (x, x) .$$

We view the middle of the cylinder $] - R/2, R/2[\times Y$ as a subset of $\mathbb{R} \times Y$. Let $D_{\mathbb{R}Y}^F$ be the Hodge-de Rham operator acting on $\Omega^\bullet(\mathbb{R} \times Y, F)$. Let ι be the involution on $\mathbb{R} \times Y$ sending (u, y) to $(-u, y)$. For $x \in (] - R/2, R/2[\times Y) \cap Z_{j,R}$ ($j = 1, 2$), we have

$$(5.38) \quad F_{t,\varsigma} \left(\sqrt{t} D_{Z_{j,R}}^F \right) (x, x) = F_{t,\varsigma} \left(\sqrt{t} D_{\mathbb{R}Y}^F \right) (x, x) + (-1)^j F_{t,\varsigma} \left(\sqrt{t} D_{\mathbb{R}Y}^F \right) (x, \iota x) .$$

As a consequence, for $x \in] - R/2, R/2[\times Y \cap Z_{1,R} =] - R/2, 0] \times Y$, we have

$$(5.39) \quad \begin{aligned} & F_{t,\varsigma} \left(\sqrt{t} D_{Z_{1,R}}^F \right) (x, x) + \iota^* F_{t,\varsigma} \left(\sqrt{t} D_{Z_{2,R}}^F \right) (\iota x, \iota x) \\ &= F_{t,\varsigma} \left(\sqrt{t} D_{Z_R}^F \right) (x, x) + \iota^* F_{t,\varsigma} \left(\sqrt{t} D_{Z_R}^F \right) (\iota x, \iota x) \in \text{End} \left(\Lambda^\bullet(T^*Z_{j,R}) \otimes F \right)_x . \end{aligned}$$

By (5.32), $\Theta_R(t)$ can be decomposed to the contributions of $F_{t,\varsigma}$ and $G_{t,\varsigma}$. By (5.37) and (5.39), the contribution of $F_{t,\varsigma}$ to (5.29) vanishes identically. By (5.36), the contribution of $G_{t,\varsigma}$ to (5.29) together with its derivative at $s = 0$ are $\mathcal{O}(e^{-R^{\varepsilon/2}})$ -small. For $\Theta_R^*(t)$, the same argument works. This terminates the proof of (5.30). \square

5.3. Large time contribution and proof of Theorem 0.1. By (5.1) and (5.15), we have

$$(5.40) \quad \sum_{j=0}^2 (-1)^{(j-1)(j-2)/2} \left(\zeta_{j,R}^L(s) - \zeta_{*,j,R}^L(s) \right) = -\frac{1}{\Gamma(s)} \int_{R^{2-\varepsilon}}^{\infty} t^{s-1} (\Theta_R(t) - \Theta_R^*(t)) dt .$$

Let $\kappa \in]\varepsilon, 1[$. Let $\Theta_R^I(t)$ (resp. $\Theta_R^{II}(t)$) be the contribution to $\Theta_R(t)$ by eigenvalues of $D_{Z_{j,R}}^{F,2}$ ($j = 0, 1, 2$) less than (resp. greater than or equal to) $R^{-2+\kappa}$. We define $\Theta_R^{*,I}(t)$ and $\Theta_R^{*,II}(t)$ in the same way.

Proposition 5.6. *As $R \rightarrow \infty$, we have*

$$(5.41) \quad \int_{R^{2-\varepsilon}}^{\infty} \Theta_R^{II}(t) \frac{dt}{t} = \mathcal{O} \left(e^{-\frac{1}{2} R^{\kappa-\varepsilon}} \right) , \quad \int_{R^{2-\varepsilon}}^{\infty} \Theta_R^{*,II}(t) \frac{dt}{t} = \mathcal{O} \left(e^{-\frac{1}{2} R^{\kappa-\varepsilon}} \right) .$$

Proof. Let $\{\lambda_k\}_k$ be the set of eigenvalues of $D_{Z_{j,R}}^{F,2}$ ($j = 0, 1, 2$) such that $\lambda_k \geq R^{-2+\kappa}$. Let $n = \dim Z$. Then for R large and $t \geq R^{2-\varepsilon}$, we have

$$(5.42) \quad \begin{aligned} |\Theta_R^I(t)| &\leq n \sum_k e^{-t\lambda_k} \leq n e^{-(t-1)R^{-2+\kappa}} \sum_k e^{-\lambda_k} \\ &\leq n e^{-(t-1)R^{-2+\kappa}} \sum_{j=0}^2 \text{Tr} \left[\exp \left(-D_{Z_{j,R}}^{F,2} \right) \right]. \end{aligned}$$

Let $\exp \left(-D_{Z_{j,R}}^{F,2} \right) (x, y)$ ($x, y \in Z_{j,R}$) be the smooth kernel of the operator $\exp \left(-D_{Z_{j,R}}^{F,2} \right)$ with respect to the volume form induced by the Riemannian metric on $Z_{j,R}$. Following the same argument as (5.36), there exists $C > 0$ such that for any $x, y \in Z_{j,R}$,

$$(5.43) \quad \left| \exp \left(-D_{Z_{j,R}}^{F,2} \right) (x, y) \right| \leq C.$$

As a consequence, there exist $a, b > 0$, such that

$$(5.44) \quad \text{Tr} \left[\exp \left(-D_{Z_{j,R}}^{F,2} \right) \right] \leq a \text{Vol}(Z_{j,R}) \leq bR, \quad \text{for } j = 0, 1, 2.$$

By (5.42) and (5.44), we get the first estimate in (5.41). The second one can be established in the same way. \square

Proposition 5.7. *As $R \rightarrow \infty$, we have*

$$(5.45) \quad \int_{R^{2-\varepsilon}}^{\infty} \left(\Theta_R^I(t) - \Theta_R^{*,I}(t) \right) \frac{dt}{t} = \mathcal{O}(R^{\kappa-1})$$

Proof. For $\lambda > 0$, we denote

$$(5.46) \quad e_R(\lambda) = \int_{R^{2-\varepsilon}}^{\infty} e^{-t\lambda} \frac{dt}{t} = \int_{R^{2-\varepsilon}\lambda}^{\infty} e^{-t} \frac{dt}{t}.$$

By splitting the integral to $\int_1^{\infty} + \int_{R^{2-\varepsilon}\lambda}^1$ (if $R^{2-\varepsilon}\lambda \leq 1$), we have

$$(5.47) \quad |e_R(\lambda)| \leq 1 + \max \{ -\log(R^{2-\varepsilon}\lambda), 0 \}, \quad |e_R'(\lambda)| \leq \lambda^{-1}.$$

For any finite set (with multiplicity) $\Lambda \subseteq \mathbb{R}$, we denote

$$(5.48) \quad e_R[\Lambda] = \sum_{\lambda \in \Lambda} e_R(\lambda).$$

Then

$$(5.49) \quad \begin{aligned} &\int_{R^{2-\varepsilon}}^{\infty} \left(\Theta_R^I(t) - \Theta_R^{*,I}(t) \right) \frac{dt}{t} \\ &= \sum_{j=0}^2 \sum_p (-1)^{(j-1)(j-2)/2+p} p \left\{ e_R \left[\text{Sp} \left(D_{Z_{j,R}}^{F,2,(p)} \right) \cap]0, R^{\kappa-2}[\right] - e_R \left[\text{Sp} \left(D_{I_{j,R}}^{2,(p)} \right) \cap]0, R^{\kappa-2}[\right] \right\}. \end{aligned}$$

We will show that

$$(5.50) \quad e_R \left[\text{Sp} \left(D_{Z_R}^{F,2,(p)} \right) \cap]0, R^{\kappa-2}[\right] - e_R \left[\text{Sp} \left(D_{I_R}^{2,(p)} \right) \cap]0, R^{\kappa-2}[\right] = \mathcal{O}(R^{\kappa-1}).$$

The other terms can be estimated in the same way, and (5.45) follows.

Recall that Λ_R^p is defined in (3.147). By Theorem 3.19, we have

$$(5.51) \quad e_R \left[\text{Sp} \left(D_{Z_R}^{F,2,(p)} \right) \cap]0, R^{\kappa-2}[\right] = \sum_{\rho \in \Lambda_R^p, 0 < |\rho| < R^{\kappa/2-1}} e_R(\rho^2) + \mathcal{O}(e^{-cR}).$$

Recall that $\Lambda_R^{*,p}$ is defined in (5.24). By (5.25), we have

$$(5.52) \quad e_R \left[\text{Sp} \left(D_{I_R}^{2,(p)} \right) \cap]0, R^{\kappa-2}[\right] = \sum_{\lambda \in \Lambda_R^{*,p}, 0 < |\lambda| < R^{\kappa/2-1}} e_R(\lambda^2).$$

By Appendix Proposition 8.3 and (5.47), we have

$$(5.53) \quad \sum_{\rho \in \Lambda_R^p, 0 < |\rho| < R^{-1+\kappa/2}} e_R(\rho^2) - \sum_{\lambda \in \Lambda_R^{*,p}, 0 < |\lambda| < R^{-1+\kappa/2}} e_R(\lambda^2) = \mathcal{O}(R^{\kappa-1}).$$

By (5.51), (5.52) and (5.53), we get (5.50). \square

Theorem 5.8. *As $R \rightarrow \infty$, we have*

$$(5.54) \quad \sum_{j=0}^2 (-1)^{(j-1)(j-2)/2} \left(\zeta_{j,R}^{L'}(0) - \zeta_{*,j,R}^{L'}(0) \right) = \mathcal{O}(R^{\kappa-1}).$$

Proof. We combine Proposition 5.6, 5.7. \square

Proof of Theorem 0.1 : We combine Proposition 5.4 and Theorem 5.5, 5.8. \square

6. ASYMPTOTICS OF THE L^2 -METRICS ON MAYER-VIETORIS SEQUENCE

In this section, we prove Theorem 0.2.

We use the notations and assumptions of §3.1 and §3.2.

In §6.1, we construct a filtration of the Mayer-Vietoris sequence. More precisely, we extend the Mayer-Vietoris sequence to a commutative diagram with exact rows and columns. Moreover, we construct another commutative diagram (6.16), which is isomorphic to the original one. In §6.2, every object in diagram (6.16) is equipped with a metric (depending on R). We study the asymptotic behavior of these metrics as $R \rightarrow \infty$. In §6.3, we study the asymptotic behavior of the maps in diagram (6.16) as $R \rightarrow \infty$. In §6.4, with the help of diagram (6.16), we prove Theorem 0.2.

6.1. A filtration of the Mayer-Vietoris sequence. Recall that (F, ∇^F) is a flat vector bundle on Z , and $Y \subseteq Z$ is a hypersurface cutting Z into Z_1, Z_2 . For $R \geq 0$, we constructed $Z_{j,R}$ ($j = 1, 2$) (resp. Z_R) from Z_j (resp. Z) by attaching a cylinder of length R (resp. $2R$), and extended F to a flat vector bundle on Z_R .

The maps $\varphi_{j,R} : Z_{j,R} \rightarrow Z_j$ ($j = 1, 2$) defined in (4.7) and $\varphi_R : Z_R \rightarrow Z$ defined in (3.32) are diffeomorphisms, which induce the identifications of cohomology groups

$$(6.1) \quad \varphi_{R*} : H_{\text{bd}}^\bullet(Z_{j,R}, F) \rightarrow H_{\text{bd}}^\bullet(Z_j, F), \quad \varphi_{R*} : H^\bullet(Z_R, F) \rightarrow H^\bullet(Z, F).$$

Since these diffeomorphisms commute with the injections $Z_j \hookrightarrow Z$ and $Z_{j,R} \hookrightarrow Z_R$, we get an isomorphism of long exact sequence

$$(6.2) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & H_{\text{bd}}^p(Z_{1,R}, F) & \longrightarrow & H^p(Z_R, F) & \longrightarrow & H_{\text{bd}}^p(Z_{2,R}, F) \longrightarrow \cdots \\ & & \downarrow \varphi_{R*} & & \downarrow \varphi_{R*} & & \downarrow \varphi_{R*} \\ \cdots & \longrightarrow & H_{\text{bd}}^p(Z_1, F) & \xrightarrow{\alpha_p} & H^p(Z, F) & \xrightarrow{\beta_p} & H_{\text{bd}}^p(Z_2, F) \xrightarrow{\delta_p} \cdots \end{array}$$

where each row is the classical Mayer-Vietoris sequence (0.15).

We recall that $D_{Z_{j,R}}^F$ ($j = 1, 2$) (resp. $D_{Z_R}^F$) is the Hodge-de Rham operator (cf. (0.2)) acting on $\Omega_{\text{bd}}^\bullet(Z_{j,R}, F)$ (resp. $\Omega^\bullet(Z_R, F)$). Its kernel is denoted $\mathcal{H}_{\text{bd}}^\bullet(Z_{j,R}, F)$ (resp. $\mathcal{H}^\bullet(Z_R, F)$). We recall that $\mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F)$ ($j = 1, 2$) is defined by (2.48) and $\mathcal{H}^\bullet(Z_{12,\infty}, F)$ is defined by (3.10). We constructed in Definition 3.6, 4.4 the bijections

$$(6.3) \quad \mathcal{F}_{Z_{j,R}} : \mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F) \rightarrow \mathcal{H}_{\text{bd}}^\bullet(Z_{j,R}, F), \quad \mathcal{F}_{Z_R} : \mathcal{H}^\bullet(Z_{12,\infty}, F) \rightarrow \mathcal{H}^\bullet(Z_R, F).$$

By Theorem 1.1, $\mathcal{F}_{Z_{j,R}}$ and \mathcal{F}_{Z_R} may be viewed as maps

$$(6.4) \quad \mathcal{F}_{Z_{j,R}} : \mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F) \rightarrow H_{\text{bd}}^\bullet(Z_{j,R}, F), \quad \mathcal{F}_{Z_R} : \mathcal{H}^\bullet(Z_{12,\infty}, F) \rightarrow H^\bullet(Z_R, F).$$

Now we define the composition map

$$(6.5) \quad \begin{aligned} \widetilde{\mathcal{F}}_{Z_{j,R}} &= \varphi_{R*} \circ \mathcal{F}_{Z_{j,R}} : \mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F) \rightarrow H_{\text{bd}}^\bullet(Z_j, F), \\ \widetilde{\mathcal{F}}_{Z_R} &= \varphi_{R*} \circ \mathcal{F}_{Z_R} : \mathcal{H}^\bullet(Z_{12,\infty}, F) \rightarrow H^\bullet(Z, F). \end{aligned}$$

It is important to notice that these maps depend on R .

Recall that the inclusion $\mathcal{H}_{L^2}^\bullet(Z_{j,\infty}, F) \subseteq \mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F)$ ($j = 1, 2$) is defined in (2.49), and the inclusion $\mathcal{H}_{L^2}^\bullet(Z_{1,\infty}, F) \oplus \mathcal{H}_{L^2}^\bullet(Z_{2,\infty}, F) \subseteq \mathcal{H}^\bullet(Z_{12,\infty}, F)$ is defined in (3.13). For simplicity, we denote $\mathcal{H}_{L^2}^\bullet(Z_{1,\infty}, F) \oplus \mathcal{H}_{L^2}^\bullet(Z_{2,\infty}, F) = \mathcal{H}_{L^2}^\bullet(Z_{12,\infty}, F)$.

For R large enough, set

$$(6.6) \quad \begin{aligned} K_j^\bullet &= \widetilde{\mathcal{F}}_{Z_{j,R}}(\mathcal{H}_{L^2}^\bullet(Z_{j,\infty}, F)) \subseteq H_{\text{bd}}^\bullet(Z_j, F), \quad \text{for } j = 1, 2, \\ K_{12}^\bullet &= \widetilde{\mathcal{F}}_{Z_R}(\mathcal{H}_{L^2}^\bullet(Z_{12,\infty}, F)) \subseteq H^\bullet(Z, F). \end{aligned}$$

By Proposition 3.3 and Proposition 4.2, K_1^\bullet , K_2^\bullet and K_{12}^\bullet do not depend on R . We define the following commutative diagram with exact rows

$$(6.7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}_{L^2}^p(Z_{1,\infty}, F) & \longrightarrow & \mathcal{H}_{L^2}^p(Z_{12,\infty}, F) & \longrightarrow & \mathcal{H}_{L^2}^p(Z_{2,\infty}, F) \longrightarrow 0 \\ & & \downarrow \widetilde{\mathcal{F}}_{Z_{1,R}} & & \downarrow \widetilde{\mathcal{F}}_{Z_R} & & \downarrow \widetilde{\mathcal{F}}_{Z_{2,R}} \\ 0 & \longrightarrow & K_1^p & \dashrightarrow & K_{12}^p & \dashrightarrow & K_2^p \longrightarrow 0 \end{array}$$

where the first row consists of canonical injection/projection maps. Still, by Proposition 3.3 and Proposition 4.2, diagram (6.7) is independent to R .

Set

$$(6.8) \quad L_{j,\text{bd}}^\bullet = H_{\text{bd}}^\bullet(Z_j, F)/K_j^\bullet, \quad L_{12}^\bullet = H^\bullet(Z, F)/K_{12}^\bullet.$$

Proposition 6.1. *We have the following commutative diagram with exact rows and columns*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & K_1^p & \longrightarrow & K_{12}^p & \longrightarrow & K_2^p \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 (6.9) \quad \cdots & \longrightarrow & H_{\text{bd}}^p(Z_1, F) & \xrightarrow{\alpha_p} & H^p(Z, F) & \xrightarrow{\beta_p} & H_{\text{bd}}^p(Z_2, F) \xrightarrow{\delta_p} \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \dashrightarrow & L_{1,\text{bd}}^p & \dashrightarrow^{\bar{\alpha}_p} & L_{12}^p & \dashrightarrow^{\bar{\beta}_p} & L_{2,\text{bd}}^p \dashrightarrow^{\bar{\delta}_p} \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where the maps $K_1^p \rightarrow K_{12}^p$ and $K_{12}^p \rightarrow K_2^p$ are defined by (6.7), the map $K_2^p \rightarrow K_1^{p+1}$ is zero, the second row is the classical Mayer-Vietoris sequence (0.15), and the vertical maps are canonical injection/projection maps.

Proof. We show that the upper left square commutes. It is equivalent to show that for any $\omega \in \mathcal{H}_{L^2}^p(Z_{1,\infty}, F)$, we have

$$(6.10) \quad \alpha_p([\mathcal{F}_{Z_{1,R}}(\omega, 0)]) = [\mathcal{F}_{Z_R}(\omega, 0, 0)] \in H^p(Z, F).$$

We recall that F_{Z_R} is defined in (3.29) and $F_{Z_{1,R}}$ is defined (4.5). By their definitions, we have

$$(6.11) \quad F_{Z_R}(\omega, 0, 0)|_{Z_{1,R}} = F_{Z_{1,R}}(\omega, 0), \quad F_{Z_R}(\omega, 0, 0)|_{Z_{2,R}} = 0.$$

And, by (3.48) and (4.22), we have

$$\begin{aligned}
 (6.12) \quad [\mathcal{F}_{Z_{1,R}}(\omega, 0)] &= [F_{Z_{1,R}}(\omega, 0)] \in H_{\text{bd}}^p(Z_1, F), \\
 [\mathcal{F}_{Z_R}(\omega, 0, 0)] &= [F_{Z_R}(\omega, 0, 0)] \in H^p(Z, F).
 \end{aligned}$$

By Proposition 1.2 and (6.11), we have

$$(6.13) \quad \alpha_p([F_{Z_{1,R}}(\omega, 0)]) = [F_{Z_R}(\omega, 0, 0)] \in H^p(Z, F).$$

Then (6.10) follows from (6.12) and (6.13).

Following the same procedure, we can show that the upper right square commutes and $\delta_p(K_2^p) = 0$. Thus, we get the commutativity between the first and second rows.

The rests can be done by direct diagram chasing arguments. \square

Let \mathcal{L}_j^\bullet ($j = 1, 2$) be the set of limiting values of $\mathcal{H}^\bullet(Z_{j,\infty}, F)$, which is defined by (2.43). Let $\mathcal{L}_{j,\text{abs/rel}}^\bullet$ be the absolute/relative component of \mathcal{L}_j^\bullet , which is defined by (2.46). Still, we use the convention $\mathcal{L}_{1,\text{bd}}^\bullet = \mathcal{L}_{1,\text{rel}}^\bullet$ and $\mathcal{L}_{2,\text{bd}}^\bullet = \mathcal{L}_{1,\text{abs}}^\bullet$.

We define, for $j = 1, 2$, the following commutative diagram with exact rows

$$(6.14) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}_{L^2}^p(Z_{j,\infty}, F) & \longrightarrow & \mathcal{H}_{\text{bd}}^p(Z_{j,\infty}, F) & \longrightarrow & \mathcal{L}_{j,\text{bd}}^p \longrightarrow 0 \\ & & \downarrow \widetilde{\mathcal{F}}_{Z_{j,R}} & & \downarrow \widetilde{\mathcal{F}}_{Z_{j,R}} & & \downarrow \overline{\mathcal{F}}_{Z_{j,R}} \\ 0 & \longrightarrow & K_j^p & \longrightarrow & H_{\text{bd}}^p(Z_j, F) & \longrightarrow & L_{j,\text{bd}}^p \longrightarrow 0 \end{array}$$

where the first row is defined by (2.49), the second row consists of canonical injection/projection maps. We define the following commutative diagram with exact rows

$$(6.15) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}_{L^2}^p(Z_{12,\infty}, F) & \longrightarrow & \mathcal{H}^p(Z_{12,\infty}, F) & \longrightarrow & \mathcal{L}_1^p \cap \mathcal{L}_2^p \longrightarrow 0 \\ & & \downarrow \widetilde{\mathcal{F}}_{Z_R} & & \downarrow \widetilde{\mathcal{F}}_{Z_R} & & \downarrow \overline{\mathcal{F}}_{Z_R} \\ 0 & \longrightarrow & K_{12}^p & \longrightarrow & H^p(Z, F) & \longrightarrow & L_{12}^p \longrightarrow 0 \end{array}$$

where the first row is defined by (3.13), the second row consists of canonical injection/projection maps.

By (6.9), (6.14) and (6.15), we get the following commutative diagram with exact rows and columns, which is the analytic counterpart of (6.9),

$$(6.16) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathcal{H}_{L^2}^p(Z_{1,\infty}, F) & \longrightarrow & \mathcal{H}_{L^2}^p(Z_{12,\infty}, F) & \longrightarrow & \mathcal{H}_{L^2}^p(Z_{2,\infty}, F) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathcal{H}_{\text{bd}}^p(Z_{1,\infty}, F) & \xrightarrow{\alpha_p(R)} & \mathcal{H}^p(Z_{12,\infty}, F) & \xrightarrow{\beta_p(R)} & \mathcal{H}_{\text{bd}}^p(Z_{2,\infty}, F) \xrightarrow{\delta_p(R)} \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathcal{L}_{1,\text{bd}}^p & \xrightarrow{\bar{\alpha}_p(R)} & \mathcal{L}_1^p \cap \mathcal{L}_2^p & \xrightarrow{\bar{\beta}_p(R)} & \mathcal{L}_{2,\text{bd}}^p \xrightarrow{\bar{\delta}_p(R)} \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

where the first row consists of canonical injection/projection maps, the columns are defined by (2.49) and (3.13), and

$$(6.17) \quad \begin{aligned} \alpha_p(R) &= \left(\widetilde{\mathcal{F}}_{Z_R} \right)^{-1} \circ \alpha_p \circ \widetilde{\mathcal{F}}_{Z_{1,R}}, & \bar{\alpha}_p(R) &= \left(\overline{\mathcal{F}}_{Z_R} \right)^{-1} \circ \bar{\alpha}_p \circ \overline{\mathcal{F}}_{Z_{1,R}}, \\ \beta_p(R) &= \left(\widetilde{\mathcal{F}}_{Z_{2,R}} \right)^{-1} \circ \beta_p \circ \widetilde{\mathcal{F}}_{Z_R}, & \bar{\beta}_p(R) &= \left(\overline{\mathcal{F}}_{Z_{2,R}} \right)^{-1} \circ \bar{\beta}_p \circ \overline{\mathcal{F}}_{Z_R}, \\ \delta_p(R) &= \left(\widetilde{\mathcal{F}}_{Z_{1,R}} \right)^{-1} \circ \delta_p \circ \widetilde{\mathcal{F}}_{Z_{2,R}}, & \bar{\delta}_p(R) &= \left(\overline{\mathcal{F}}_{Z_{1,R}} \right)^{-1} \circ \bar{\delta}_p \circ \overline{\mathcal{F}}_{Z_{2,R}}. \end{aligned}$$

6.2. Asymptotics of the L^2 -metrics. We begin by equipping the spaces in the second row of diagram (6.16) with metrics.

We recall that the metric $\|\cdot\|_{\mathcal{H}^\bullet(Z_{12,\infty},F),R}$ on $\mathcal{H}^\bullet(Z_{12,\infty},F)$ is defined by (3.14). Let $\mathcal{F}_{Z_R}^* (\|\cdot\|_{H^\bullet(Z_R,F)})$ be another metric on $\mathcal{H}^\bullet(Z_{12,\infty},F)$, which is the pull-back of the L^2 -metric (defined in §0.4) $\|\cdot\|_{H^\bullet(Z_R,F)}$ on $H^\bullet(Z_R,F)$ via \mathcal{F}_{Z_R} (cf. Definition 3.6).

We recall that the metric $\|\cdot\|_{\mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty},F),R}$ ($j = 1, 2$) on $\mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty},F)$ is defined by (4.19). Let $\mathcal{F}_{Z_{j,R}}^*(\|\cdot\|_{H_{\text{bd}}^\bullet(Z_{j,R},F)})$ be another metric on $\mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty},F)$, which is the pull-back of the L^2 -metric $\|\cdot\|_{H_{\text{bd}}^\bullet(Z_{j,R},F)}$ on $H_{\text{bd}}^\bullet(Z_{j,R},F)$ via $\mathcal{F}_{Z_{j,R}}$ (cf. Definition 4.4).

Proposition 6.2. *There exists $c > 0$, such that, as $R \rightarrow +\infty$, we have*

$$(6.18) \quad \begin{aligned} \mathcal{F}_{Z_{j,R}}^*(\|\cdot\|_{H_{\text{bd}}^\bullet(Z_{j,R},F)}) &= \|\cdot\|_{\mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty},F),R} + \mathcal{O}(e^{-cR}), \quad \text{for } j = 1, 2, \\ \mathcal{F}_{Z_R}^*(\|\cdot\|_{H^\bullet(Z_R,R)}) &= \|\cdot\|_{\mathcal{H}^\bullet(Z_{12,\infty},F),R} + \mathcal{O}(e^{-cR}). \end{aligned}$$

Proof. The first identity is a direct consequence of Proposition 4.3, 4.5. The second identity is a direct consequence of Proposition 3.4, 3.7. \square

Now we equip the spaces in the third row of diagram (6.16) with metrics.

Let $\|\cdot\|_{\mathcal{L}_1^\bullet \cap \mathcal{L}_2^\bullet, R}$ be the quotient metric on $\mathcal{L}_1^\bullet \cap \mathcal{L}_2^\bullet$ induced by $\|\cdot\|_{\mathcal{H}^\bullet(Z_{12,\infty},F),R}$ via the vertical map $\mathcal{H}^\bullet(Z_{12,\infty},F) \rightarrow \mathcal{L}_1^\bullet \cap \mathcal{L}_2^\bullet$ in diagram (6.16). Let $\|\cdot\|_{\mathcal{L}_1^\bullet \cap \mathcal{L}_2^\bullet}$ be another metric on $\mathcal{L}_1^\bullet \cap \mathcal{L}_2^\bullet$, which is induced by the L^2 -metric $\|\cdot\|_Y$ on $\mathcal{H}^\bullet(Y, F[du])$ via the inclusion $\mathcal{L}_1^\bullet \cap \mathcal{L}_2^\bullet \subseteq \mathcal{H}^\bullet(Y, F[du])$ (cf. (2.43)).

In the same way, we define metrics $\|\cdot\|_{\mathcal{L}_{j,\text{bd}}^\bullet, R}$ and $\|\cdot\|_{\mathcal{L}_{j,\text{bd}}^\bullet}$ on $\mathcal{L}_{j,\text{bd}}^\bullet$.

Proposition 6.3. *As $R \rightarrow +\infty$, we have*

$$(6.19) \quad \begin{aligned} \|\cdot\|_{\mathcal{L}_{j,\text{bd}}^\bullet, R}^2 &= R \|\cdot\|_{\mathcal{L}_{j,\text{bd}}^\bullet}^2 + \mathcal{O}(1), \quad \text{for } j = 1, 2, \\ \|\cdot\|_{\mathcal{L}_1^\bullet \cap \mathcal{L}_2^\bullet, R}^2 &= 2R \|\cdot\|_{\mathcal{L}_1^\bullet \cap \mathcal{L}_2^\bullet}^2 + \mathcal{O}(1). \end{aligned}$$

Proof. We only prove the first one with $j = 2$. The others can be proved in the same way.

We recall that $\mathcal{H}_{\text{bd}}^\bullet(Z_{2,\infty}, F)$ is defined by (2.48). By the definition of quotient metric, for any $\hat{\omega} \in \mathcal{L}_{2,\text{bd}}^\bullet$, we have

$$(6.20) \quad \|\hat{\omega}\|_{\mathcal{L}_{2,\text{bd}}^\bullet, R}^2 = \inf_{(\omega, \hat{\omega}) \in \mathcal{H}_{\text{bd}}^\bullet(Z_{2,\infty}, F)} \|(\omega, \hat{\omega})\|_{\mathcal{H}_{\text{bd}}^\bullet(Z_{2,\infty}, F), R}^2.$$

We recall that $I_{2,\infty}Y \subseteq Z_{2,\infty}$ is its cylinder part, defined in §3.1. On $I_{2,\infty}Y$, let $\omega = \omega^{\text{zm}} + \omega^{\text{nz}}$ be the decomposition of ω into zero-mode and zero-mode free parts, defined in (2.16). Recall that $\pi_Y : I_{2,\infty}Y \rightarrow Y$ is the natural projection, and we have $\pi_Y^* \hat{\omega} = \omega^{\text{zm}}$. As a consequence, we have

$$(6.21) \quad \|\omega^{\text{zm}}\|_{I_{2,R}Y}^2 = R \|\hat{\omega}\|_Y^2 = R \|\hat{\omega}\|_{\mathcal{L}_{2,\text{bd}}^\bullet}^2,$$

where $I_{2,R}Y \subseteq Z_{2,R}$ is the cylinder part of $Z_{2,R}$, also defined in §3.1. Thus

$$(6.22) \quad \|(\omega, \hat{\omega})\|_{\mathcal{H}_{\text{bd}}^\bullet(Z_{2,\infty}, F), R}^2 - R \|\hat{\omega}\|_{\mathcal{L}_{2,\text{bd}}^\bullet}^2 = \|\omega\|_{Z_{2,R}}^2 - \|\omega^{\text{zm}}\|_{I_{2,R}Y}^2 = \|\omega\|_{Z_{2,0}}^2 + \|\omega^{\text{nz}}\|_{I_{2,R}Y}^2.$$

In particular, we have

$$(6.23) \quad \|(\omega, \hat{\omega})\|_{\mathcal{H}_{\text{bd}}^\bullet(Z_{2,\infty}, F), R}^2 \geq R \|\hat{\omega}\|_{\mathcal{L}_{2,\text{bd}}^\bullet}^2.$$

By (6.20), (6.22) and (6.23), it is sufficient to show that there exists $C > 0$, such that for any $\hat{\omega} \in \mathcal{L}_{2,\text{bd}}^\bullet$, there exists $(\omega, \hat{\omega}) \in \mathcal{H}_{\text{bd}}^\bullet(Z_{2,\infty}, F)$, such that, for any $R > 0$,

$$(6.24) \quad \|\omega\|_{Z_{2,0}}^2 + \|\omega^{\text{nz}}\|_{I_{2,R}Y}^2 \leq C \|\hat{\omega}\|_{\mathcal{L}_{2,\text{bd}}^\bullet}^2.$$

In the rest of the proof, we choose ω such that $(\omega, \hat{\omega}) \in \mathcal{H}_{\text{bd}}^\bullet(Z_{2,\infty}, F)$ and ω is a generalized eigensection of $D_{Z_{2,\infty}}^F$ with eigenvalue $\lambda = 0$. The existence and uniqueness of such a generalized eigensection comes from Remark 2.8. By (2.33), there exists $C_1 > 0$, such that for any $\hat{\omega}$ and its associated generalized eigensection ω , we have

$$(6.25) \quad \|\omega\|_{Z_{2,0}}^2 \leq C_1 \|\hat{\omega}\|_Y^2 = C_1 \|\hat{\omega}\|_{\mathcal{L}_{2,\text{bd}}^\bullet}^2.$$

By Lemma 2.1 and (3.43) with $Z_{1,0}$ replaced by $Z_{2,0}$, there exists $C_2 > 0$, such that for any generalized eigensection ω with eigenvalue $\lambda = 0$, we have

$$(6.26) \quad \|\omega^{\text{nz}}\|_{I_{2,R}Y}^2 \leq \|\omega^{\text{nz}}\|_{I_{2,\infty}Y}^2 \leq C_2 \|\omega\|_{Z_{2,0}}^2.$$

By (6.25)-(6.26), we get (6.24). \square

6.3. Asymptotics of the horizontal maps. We begin by considering the second row of diagram (6.16).

We recall that the operators $du \wedge, i_{\frac{\partial}{\partial u}}$ and $c(\frac{\partial}{\partial u})$ on $\Omega^\bullet(Y, F[du])$ or $\mathcal{H}^\bullet(Y, F[du])$ are defined in (2.4).

In the sequel, $\mathcal{O}(e^{-cR})$ means a number bounded by Ce^{-cR} , with $c, C > 0$ determined by Z_1, Z_2 and F and independent to other objects concerned in the formula involving $\mathcal{O}(e^{-cR})$. We use the notations $\mathcal{O}(R^{-1}), \mathcal{O}(R^{-2})$, etc., in the same way.

Proposition 6.4. *For $(\omega, \hat{\omega}) \in \mathcal{H}_{\text{bd}}^p(Z_{1,\infty}, F)$ and $(\mu_1, \mu_2, \hat{\mu}) \in \mathcal{H}^p(Z_{12,\infty}, F)$, we have*

$$(6.27) \quad \begin{aligned} & \langle \alpha_p(R)(\omega, \hat{\omega}), (\mu_1, \mu_2, \hat{\mu}) \rangle_{\mathcal{H}^p(Z_{12,\infty}, F), R} \\ &= \langle \omega, \mu_1 \rangle_{Z_{1,R}} + \mathcal{O}(e^{-cR}) \|\omega, \hat{\omega}\|_{\mathcal{H}_{\text{bd}}^p(Z_{1,\infty}, F)} \|\mu_1, \mu_2, \hat{\mu}\|_{\mathcal{H}^p(Z_{12,\infty}, F)}. \end{aligned}$$

For $(\omega_1, \omega_2, \hat{\omega}) \in \mathcal{H}^p(Z_{12,\infty}, F)$ and $(\mu, \hat{\mu}) \in \mathcal{H}_{\text{bd}}^p(Z_{2,\infty}, F)$, we have

$$(6.28) \quad \begin{aligned} & \langle \beta_p(R)(\omega_1, \omega_2, \hat{\omega}), (\mu, \hat{\mu}) \rangle_{\mathcal{H}_{\text{bd}}^p(Z_{2,\infty}, F), R} \\ &= \langle \omega_2, \mu \rangle_{Z_{2,R}} + \mathcal{O}(e^{-cR}) \|(\omega_1, \omega_2, \hat{\omega})\|_{\mathcal{H}^p(Z_{12,\infty}, F)} \|(\mu, \hat{\mu})\|_{\mathcal{H}_{\text{bd}}^p(Z_{2,\infty}, F)}. \end{aligned}$$

For $(\omega, \hat{\omega}) \in \mathcal{H}_{\text{bd}}^p(Z_{2,\infty}, F)$ and $(\mu, \hat{\mu}) \in \mathcal{H}_{\text{bd}}^{p+1}(Z_{1,\infty}, F)$, we have

$$(6.29) \quad \begin{aligned} & \langle \delta_p(R)(\omega, \hat{\omega}), (\mu, \hat{\mu}) \rangle_{\mathcal{H}^{p+1}(Z_{1,\infty}, F), R} \\ &= \left\langle \hat{\omega}, i_{\frac{\partial}{\partial u}} \hat{\mu} \right\rangle_Y + \mathcal{O}(e^{-cR}) \|(\omega, \hat{\omega})\|_{\mathcal{H}_{\text{bd}}^p(Z_{2,\infty}, F)} \|(\mu, \hat{\mu})\|_{\mathcal{H}_{\text{bd}}^{p+1}(Z_{1,\infty}, F)}. \end{aligned}$$

Proof. Once again, we recall that $\mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F)$ ($j = 1, 2$) is defined by (2.48) and $\mathcal{H}^\bullet(Z_{12,\infty}, F)$ is defined by (3.10).

For $(\omega, \hat{\omega}) \in \mathcal{H}_{\text{bd}}^p(Z_{1,\infty}, F)$, we denote

$$(6.30) \quad \alpha_p(R)(\omega, \hat{\omega}) = (\omega'_1, \omega'_2, \hat{\omega}') \in \mathcal{H}^p(Z_{12,\infty}, F).$$

By (6.17) and (6.30), we have

$$(6.31) \quad \alpha_p([\mathcal{F}_{Z_{1,R}}(\omega, \hat{\omega})]) = [\mathcal{F}_{Z_R}(\omega'_1, \omega'_2, \hat{\omega}')] \in H^p(Z, F).$$

Then, by Proposition 1.3, for any $(\mu_1, \mu_2, \hat{\mu}) \in \mathcal{H}^p(Z_{12,\infty}, F)$, we have

$$(6.32) \quad \langle \mathcal{F}_{Z_R}(\omega'_1, \omega'_2, \hat{\omega}'), \mathcal{F}_{Z_R}(\mu_1, \mu_2, \hat{\mu}) \rangle_{Z_R} = \langle \mathcal{F}_{Z_{1,R}}(\omega, \hat{\omega}), \mathcal{F}_{Z_R}(\mu_1, \mu_2, \hat{\mu}) \rangle_{Z_{1,R}}.$$

By Proposition 6.2, we have

$$(6.33) \quad \begin{aligned} & \langle (\omega'_1, \omega'_2, \hat{\omega}'), (\mu_1, \mu_2, \hat{\mu}) \rangle_{\mathcal{H}^p(Z_{12,\infty}, F), R} \\ &= \langle \mathcal{F}_{Z_R}(\omega'_1, \omega'_2, \hat{\omega}'), \mathcal{F}_{Z_R}(\mu_1, \mu_2, \hat{\mu}) \rangle_{Z_R} (1 + \mathcal{O}(e^{-cR})) . \end{aligned}$$

By Proposition 3.4, 3.7, 4.3, 4.5, we have

$$(6.34) \quad \begin{aligned} & \langle \mathcal{F}_{Z_{1,R}}(\omega, \hat{\omega}), \mathcal{F}_{Z_R}(\mu_1, \mu_2, \hat{\mu}) \rangle_{Z_{1,R}} \\ &= \langle \omega, \mu_1 \rangle_{Z_{1,R}} + \mathcal{O}(e^{-cR}) \|(\omega, \hat{\omega})\|_{\mathcal{H}_{\text{bd}}^p(Z_{1,\infty}, F)} \|(\mu_1, \mu_2, \hat{\mu})\|_{\mathcal{H}^p(Z_{12,\infty}, F)} . \end{aligned}$$

By (6.30) and (6.32)-(6.34), we get (6.27).

The second and third equations can be proved following the same procedure. \square

Now we consider the third row of diagram (6.16). Recall that we have defined the exact sequence (5.11) involving the same spaces. The comparison between (5.11) and the third row of diagram (6.16) is done in the following proposition.

Proposition 6.5. *As $R \rightarrow \infty$, we have*

$$(6.35) \quad \begin{aligned} \bar{\alpha}_p(R) &= \frac{1}{2} \alpha_{p,\mathcal{L}} + \mathcal{O}(R^{-1}) , \\ \bar{\beta}_p(R) &= \beta_{p,\mathcal{L}} + \mathcal{O}(R^{-1}) , \\ \bar{\delta}_p(R) &= R^{-1} \delta_{p,\mathcal{L}} + \mathcal{O}(R^{-2}) . \end{aligned}$$

Proof. We only prove the first one. The rests can be proved in the same way.

For $\hat{\omega} \in \mathcal{L}_{1,\text{bd}}^p$, by Remark 2.8, there exists $(\omega, \hat{\omega}) \in \mathcal{H}_{\text{bd}}^p(Z_{1,\infty}, F)$ such that ω is a generalized eigensection. Same as (6.30), we denote

$$(6.36) \quad \alpha_p(R)(\omega, \hat{\omega}) = (\omega'_1, \omega'_2, \hat{\omega}') \in \mathcal{H}^p(Z_{12,\infty}, F) .$$

Then, by (6.17),

$$(6.37) \quad \bar{\alpha}_p(R)(\hat{\omega}) = \hat{\omega}' .$$

We need to show that

$$(6.38) \quad \left\| \hat{\omega}' - \frac{1}{2} \alpha_{p,\mathcal{L}}(\hat{\omega}) \right\|_{\mathcal{L}_1^p \cap \mathcal{L}_2^p}^2 = \mathcal{O}(R^{-2}) \|\hat{\omega}\|_Y^2 .$$

By Proposition 6.3, it is sufficient to show that

$$(6.39) \quad \left\| \hat{\omega}' - \frac{1}{2} \alpha_{p,\mathcal{L}}(\hat{\omega}) \right\|_{\mathcal{L}_1^p \cap \mathcal{L}_2^p, R}^2 = \mathcal{O}(R^{-1}) \|\hat{\omega}\|_Y^2 .$$

Still, by Remark 2.8, there exists $(\omega''_1, \omega''_2, \hat{\omega}'') \in \mathcal{H}^p(Z_{12,\infty}, F)$ such that ω''_1 and ω''_2 are generalized eigensections and

$$(6.40) \quad \hat{\omega}'' = \frac{1}{2} \alpha_{p,\mathcal{L}}(\hat{\omega}) .$$

Since $\|\cdot\|_{\mathcal{L}_1^p \cap \mathcal{L}_2^p, R}$ is the quotient metric induced by $\|\cdot\|_{\mathcal{H}^p(Z_{12,\infty}, F), R}$, for proving (6.39), it is sufficient to show that

$$(6.41) \quad \|(\omega'_1, \omega'_2, \hat{\omega}') - (\omega''_1, \omega''_2, \hat{\omega}'')\|_{\mathcal{H}^p(Z_{12,\infty}, F), R}^2 = \mathcal{O}(R^{-1}) \|\hat{\omega}\|_Y^2 .$$

By Riesz representation theorem, it is equivalent to show that, for any $(\mu_1, \mu_2, \hat{\mu}) \in \mathcal{H}^p(Z_{12,\infty}, F)$, we have

$$(6.42) \quad \begin{aligned} & \langle (\omega'_1, \omega'_2, \hat{\omega}') - (\omega''_1, \omega''_2, \hat{\omega}''), (\mu_1, \mu_2, \hat{\mu}) \rangle_{\mathcal{H}^p(Z_{12,\infty}, F), R} \\ &= \mathcal{O}(R^{-1/2}) \|\hat{\omega}\|_Y \|(\mu_1, \mu_2, \hat{\mu})\|_{\mathcal{H}^p(Z_{12,\infty}, F), R} . \end{aligned}$$

By Proposition 6.4 and (6.36), we have

$$(6.43) \quad \begin{aligned} & \langle (\omega'_1, \omega'_2, \hat{\omega}'), (\mu_1, \mu_2, \hat{\mu}) \rangle_{\mathcal{H}^p(Z_{12,\infty}, F), R} \\ &= \langle \omega, \mu_1 \rangle_{Z_{1,R}} + \mathcal{O}(e^{-cR}) \|(\omega, \hat{\omega})\|_{\mathcal{H}_{\text{bd}}^p(Z_{1,\infty}, F)} \|(\mu_1, \mu_2, \hat{\mu})\|_{\mathcal{H}^p(Z_{12,\infty}, F)} . \end{aligned}$$

Since ω is a generalized eigensection, by (2.33), we have

$$(6.44) \quad \|(\omega, \hat{\omega})\|_{\mathcal{H}_{\text{bd}}^p(Z_{1,\infty}, F)} = \|\omega\|_{Z_{1,0}} = \mathcal{O}(1) \|\hat{\omega}\|_Y .$$

By (6.43) and (6.44), we get

$$(6.45) \quad \begin{aligned} & \langle (\omega'_1, \omega'_2, \hat{\omega}'), (\mu_1, \mu_2, \hat{\mu}) \rangle_{\mathcal{H}^p(Z_{12,\infty}, F), R} \\ &= \langle \omega, \mu_1 \rangle_{Z_{1,R}} + \mathcal{O}(e^{-cR}) \|\hat{\omega}\|_Y \|(\mu_1, \mu_2, \hat{\mu})\|_{\mathcal{H}^p(Z_{12,\infty}, F)} . \end{aligned}$$

The following is just the definition of $\langle \cdot, \cdot \rangle_{\mathcal{H}^p(Z_{12,\infty}, F), R}$ (cf. (3.14)),

$$(6.46) \quad \langle (\omega''_1, \omega''_2, \hat{\omega}''), (\mu_1, \mu_2, \hat{\mu}) \rangle_{\mathcal{H}^p(Z_{12,\infty}, F), R} = \langle \omega''_1, \mu_1 \rangle_{Z_{1,R}} + \langle \omega''_2, \mu_2 \rangle_{Z_{2,R}} .$$

Comparing (3.15), (6.42), (6.45) and (6.46), it rests to show that

$$(6.47) \quad \begin{aligned} & \langle \omega, \mu_1 \rangle_{Z_{1,R}} - \langle \omega''_1, \mu_1 \rangle_{Z_{1,R}} - \langle \omega''_2, \mu_2 \rangle_{Z_{2,R}} \\ &= \mathcal{O}(R^{-1/2}) \|\hat{\omega}\|_Y \|(\mu_1, \mu_2, \hat{\mu})\|_{\mathcal{H}^p(Z_{12,\infty}, F), R} . \end{aligned}$$

Since ω''_1, ω''_2 and ω are generalized eigensections, by using Lemma 2.1 and (2.33) in the same way as the proof of Proposition 6.3, we get

$$(6.48) \quad \begin{aligned} \langle \omega''_j, \mu_j \rangle_{Z_{j,R}} &= R \langle \hat{\omega}'', \hat{\mu} \rangle_Y + \mathcal{O}(1) \|\hat{\omega}\|_Y \|\hat{\mu}\|_Y \\ &= R \langle \hat{\omega}'', \hat{\mu} \rangle_Y + \mathcal{O}(R^{-1/2}) \|\hat{\omega}\|_Y \|(\mu_1, \mu_2, \hat{\mu})\|_{\mathcal{H}^p(Z_{12,\infty}, F), R} , \quad \text{for } j = 1, 2 , \\ \langle \omega, \mu_1 \rangle_{Z_{1,R}} &= R \langle \hat{\omega}, \hat{\mu} \rangle_Y + \mathcal{O}(1) \|\hat{\omega}\|_Y \|\hat{\mu}\|_Y \\ &= R \langle \hat{\omega}, \hat{\mu} \rangle_Y + \mathcal{O}(R^{-1/2}) \|\hat{\omega}\|_Y \|(\mu_1, \mu_2, \hat{\mu})\|_{\mathcal{H}^p(Z_{12,\infty}, F), R} . \end{aligned}$$

By (5.8) and (6.40), we have

$$(6.49) \quad \langle \hat{\omega}'', \hat{\mu} \rangle_Y = \frac{1}{2} \langle \alpha_{p,\mathcal{L}}(\hat{\omega}), \hat{\mu} \rangle_Y = \frac{1}{2} \langle \hat{\omega}, \hat{\mu} \rangle_Y .$$

By (6.48) and (6.49), we obtain (6.47). This finishes the proof of the first equation. \square

Remark 6.6. A special case of the problem studied in this subsection is considered by Müller-Strohmaier [32]. They consider the following Mayer-Vietoris sequence

$$(6.50) \quad \cdots \longrightarrow H_{\text{rel}}^p(Z_{1,R}, \mathbb{C}) \xrightarrow{\alpha_p} H_{\text{abs}}^p(Z_{1,R}, \mathbb{C}) \xrightarrow{\beta_p} H^p(Y, \mathbb{C}) \xrightarrow{\delta_p} \cdots$$

and give an asymptotic estimate of the sesquilinear form

$$(6.51) \quad H^p(Y, \mathbb{C}) \times H^p(Y, \mathbb{C}) \rightarrow \mathbb{C} ; (\phi, \varphi) \mapsto \langle \delta_p \phi, \delta_p \varphi \rangle ,$$

as $R \rightarrow \infty$ ([32, Theorem 3.3]), where $\langle \cdot, \cdot \rangle$ is the L^2 -metric on $H_{\text{rel}}^\bullet(Z_{1,R}, \mathbb{C})$.

6.4. Torsion of the Mayer-Vietoris sequence : proof of Theorem 0.2. Firstly, we state a technical lemma.

For $A : V \rightarrow W$ a linear map between Hermitian vector spaces with the same dimension, we denote by $\det(A)$ the determinant of the matrix of A under any orthogonal bases, which is well-defined up to $U(1) := \{z \in \mathbb{C} : |z| = 1\}$.

We recall that $\det^*(\cdot)$ is defined by (0.24).

Lemma 6.7. *Let V be a Hermitian vector space, $H_1, H_2 \subseteq V$ two subspaces. Let P_j be the orthogonal projection to H_j for $j = 1, 2$. We have*

$$(6.52) \quad |\det(P_1|_{\text{Im}(P_2 P_1)})| = |\det(P_2|_{\text{Im}(P_1 P_2)})| = \det^*(\text{Id} - P_1 - P_2 + P_1 P_2 + P_2 P_1)^{\frac{1}{4}} .$$

Proof. We claim that there exists an orthogonal decomposition $V = \bigoplus_k V_k$ such that $\dim V_k \leq 2$ and $H_j = \bigoplus_k (V_k \cap H_j)$ for $j = 1, 2$. Once the claim is proved, we may suppose that $\dim V \leq 2$. Then the only non trivial case is $\dim V = 2$ and $\dim H_1 = \dim H_2 = 1$. We may suppose that

$$(6.53) \quad V = \mathbb{C}^2 , \quad H_1 = \mathbb{C}(1, 0) , \quad H_2 = \mathbb{C}(\cos \theta, \sin \theta) , \quad \text{with } 0 \leq \theta \leq \frac{\pi}{2} .$$

We have $|\det(P_1|_{\text{Im}(P_2 P_1)})| = |\det(P_2|_{\text{Im}(P_1 P_2)})| = \cos \theta$, and

$$(6.54) \quad P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \quad P_2 = \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix} .$$

Then (6.52) follows from a direct calculation.

Now we prove the claim. The operator $P_1 P_2 P_1$ (resp. $P_2 P_1 P_2$) acting on H_1 (resp. H_2) is self-adjoint, let

$$(6.55) \quad H_1 = \bigoplus_{0 \leq \lambda \leq 1} H_1^\lambda , \quad H_2 = \bigoplus_{0 \leq \lambda \leq 1} H_2^\lambda$$

be the associated spectral decompositions, i.e.,

$$(6.56) \quad P_1 P_2 P_1|_{H_1^\lambda} = \lambda \text{Id} , \quad P_2 P_1 P_2|_{H_2^\lambda} = \lambda \text{Id} .$$

We have

$$(6.57) \quad H_1^1 = H_2^1 = H_1 \cap H_2 , \quad H_1^0 = H_1 \cap H_2^\perp , \quad H_2^0 = H_2 \cap H_1^\perp .$$

We get the orthogonal decomposition

$$(6.58) \quad V = (H_1 + H_2)^\perp \oplus (H_1 \cap H_2) \oplus (H_1 \cap H_2^\perp) \oplus (H_2 \cap H_1^\perp) \oplus \bigoplus_{0 < \lambda < 1} (H_1^\lambda + H_2^\lambda) ,$$

which is invariant under the actions of P_1 and P_2 . The problem decomposes to each block. In $H_1 \cap H_2$, the vector spaces in question are both the whole space. We take $(e_j)_j$ an orthogonal basis of $H_1 \cap H_2$ and let $V_j = \mathbb{C}e_j$. For similar reasons, the claim is true for $(H_1 + H_2)^\perp$, $H_1 \cap H_2^\perp$ and $H_2 \cap H_1^\perp$. For $H_1^\lambda + H_2^\lambda$ with $0 < \lambda < 1$, let $(v_j)_{1 \leq j \leq r}$ be an orthogonal basis of H_1^λ , let V_j be the vector space spanned by $\{v_j, P_2 v_j\}$. These V_j satisfy the desired condition. \square

We briefly recall some properties of torsion (cf. [6, §1a]), which are of constant use in this subsection. For a finite acyclic complex (V^\bullet, ∂) of Hermitian vector spaces, we denote its torsion (cf. (0.14)) by $\mathcal{T}(V^\bullet, \partial)$.

- Let $(V^\bullet[n], \partial)$ be the n -th right-shift of (V^\bullet, ∂) , i.e., $V^k[n] = V^{k-n}$, then

$$(6.59) \quad \mathcal{T}(V^\bullet[n], \partial) = (\mathcal{T}(V^\bullet, \partial))^{(-1)^n}.$$

- If (V^\bullet, ∂) is the direct sum of two complexes $(V_1^\bullet, \partial_1)$ and $(V_2^\bullet, \partial_2)$, then

$$(6.60) \quad \mathcal{T}(V^\bullet, \partial) = \mathcal{T}(V_1^\bullet, \partial_1) \cdot \mathcal{T}(V_2^\bullet, \partial_2).$$

- For a short acyclic complex

$$(6.61) \quad (V^\bullet, \partial) : 0 \rightarrow V^1 \rightarrow V^2 \rightarrow 0,$$

let A be the matrix of $\partial : V^1 \rightarrow V^2$ with respect to any orthogonal bases, then

$$(6.62) \quad \mathcal{T}(V^\bullet, \partial) = |\det(A)|.$$

Let $T_{\mathcal{L}}$ be the torsion of the sequence (5.11) equipped with metrics $\|\cdot\|_{\mathcal{L}_{j,\text{bd}}^\bullet}$ ($j = 1, 2$) and $\|\cdot\|_{\mathcal{L}_1^\bullet \cap \mathcal{L}_2^\bullet}$. Before proving Theorem 0.2, we calculate $T_{\mathcal{L}}$.

We recall that $\mathcal{L}_{j,\text{abs}}^\bullet \subseteq \mathcal{H}^\bullet(Y, F)$ ($j = 1, 2$) is the absolute component of $\mathcal{L}_j^\bullet \subseteq \mathcal{H}^\bullet(Y, F[du])$, defined by (2.46). Let $\mathcal{L}_{j,\text{abs}}^{\bullet,\perp} \subseteq \mathcal{H}^\bullet(Y, F)$ be its orthogonal complement with respect to the L^2 -metric on $\mathcal{H}^\bullet(Y, F)$. Let $S_j^p \in \text{End}(\mathcal{H}^p(Y, F))$, such that

$$(6.63) \quad S_j^p = \text{Id}_{\mathcal{L}_{j,\text{abs}}^p} - \text{Id}_{\mathcal{L}_{j,\text{abs}}^{p,\perp}}.$$

By identifying $\mathcal{H}^p(Y, F)$ to $\mathcal{H}^p(Y, F)du$ via the left multiplication $du \wedge$, defined by (2.4), S_j^p also acts on $\mathcal{H}^p(Y, F)du$.

We recall that $C_j(\lambda) \in \text{End}(\mathcal{H}^\bullet(Y, F[du]))$ ($j = 1, 2$) is the scattering matrix associated with $\Omega^\bullet(Z_{j,\infty}, F)$ (cf. §3.2). We recall that $C_j = C_j(0)$ and C_j^p is its restriction to $\mathcal{H}^p(Y, F) \oplus \mathcal{H}^{p-1}(Y, F)du$. By (2.45) and (2.46), we have

$$(6.64) \quad C_j = \begin{pmatrix} S_j^p & 0 \\ 0 & -S_j^{p-1} \end{pmatrix}.$$

Proposition 6.8. *The following identities hold*

$$(6.65) \quad \begin{aligned} T_{\mathcal{L}} &= \prod_{p=0}^{\dim Z} \det^* \left(\frac{2 - S_1^p \circ S_2^p - S_2^p \circ S_1^p}{4} \right)^{\frac{1}{4}(-1)^p} \\ &= \prod_{p=0}^{\dim Z} \det^* \left(\frac{2 - C_{12}^p - (C_{12}^p)^{-1}}{4} \right)^{\frac{1}{4}(-1)^p}. \end{aligned}$$

Proof. The sequence (5.11) is the orthogonal sum of the following two sequences

$$(6.66) \quad \begin{aligned} \cdots &\longrightarrow \mathcal{L}_{1,\text{rel}}^p \cap \mathcal{L}_{2,\text{rel}}^p \longrightarrow \mathcal{L}_1^p \cap \mathcal{L}_2^p \longrightarrow \mathcal{L}_{1,\text{abs}}^p \cap \mathcal{L}_{2,\text{abs}}^p \xrightarrow{\delta_{p,\mathcal{L}}} \cdots, \\ \cdots &\longrightarrow \mathcal{L}_{1,\text{rel}}^p \cap (\mathcal{L}_{1,\text{rel}}^p \cap \mathcal{L}_{2,\text{rel}}^p)^\perp \longrightarrow 0 \longrightarrow \mathcal{L}_{2,\text{abs}}^p \cap (\mathcal{L}_{1,\text{abs}}^p \cap \mathcal{L}_{2,\text{abs}}^p)^\perp \xrightarrow{\delta_{p,\mathcal{L}}} \cdots, \end{aligned}$$

where $\delta_{p,\mathcal{L}}$ in the line is zero map. Since the maps in the first sequence in (6.66) are canonical injection/projection maps, by (6.60) and (6.62), the first sequence in (6.66)

does not contribute to $T_{\mathcal{L}}$. And the second sequence in (6.66) splits into the short exact sequences

$$(6.67) \quad 0 \longrightarrow \mathcal{L}_{2,\text{abs}}^p \cap (\mathcal{L}_{1,\text{abs}}^p \cap \mathcal{L}_{2,\text{abs}}^p)^\perp \xrightarrow{\delta_{p,\mathcal{L}}} \mathcal{L}_{1,\text{rel}}^{p+1} \cap (\mathcal{L}_{1,\text{rel}}^{p+1} \cap \mathcal{L}_{2,\text{rel}}^{p+1})^\perp \longrightarrow 0 .$$

Using (2.46), we see that the map $i_{\frac{\partial}{\partial u}} : \mathcal{H}^p(Y, F) du \rightarrow \mathcal{H}^p(Y, F)$ send $\mathcal{L}_{1,\text{rel}}^{p+1} \cap (\mathcal{L}_{1,\text{rel}}^{p+1} \cap \mathcal{L}_{2,\text{rel}}^{p+1})^\perp$ to $\mathcal{L}_{1,\text{abs}}^{p,\perp} \cap (\mathcal{L}_{1,\text{abs}}^{p,\perp} \cap \mathcal{L}_{2,\text{abs}}^{p,\perp})^\perp$. We define the following commutative diagram with exact rows and isometric vertical maps

$$(6.68) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{L}_{2,\text{abs}}^p \cap (\mathcal{L}_{1,\text{abs}}^p \cap \mathcal{L}_{2,\text{abs}}^p)^\perp & \longrightarrow & \mathcal{L}_{1,\text{rel}}^{p+1} \cap (\mathcal{L}_{1,\text{rel}}^{p+1} \cap \mathcal{L}_{2,\text{rel}}^{p+1})^\perp & \longrightarrow & 0 \\ & & \downarrow \text{Id} & & \downarrow i_{\frac{\partial}{\partial u}} & & \\ 0 & \longrightarrow & \mathcal{L}_{2,\text{abs}}^p \cap (\mathcal{L}_{1,\text{abs}}^p \cap \mathcal{L}_{2,\text{abs}}^p)^\perp & \dashrightarrow & \mathcal{L}_{1,\text{abs}}^{p,\perp} \cap (\mathcal{L}_{1,\text{abs}}^{p,\perp} \cap \mathcal{L}_{2,\text{abs}}^{p,\perp})^\perp & \longrightarrow & 0 . \end{array}$$

By (5.10), the map in the second row in (6.68) is orthogonal projection. Furthermore, since the vertical maps are isometric, the torsions of the first and second row coincide.

Let P_p (resp. Q_p) be the orthogonal projection from $\mathcal{H}^p(Y, F)$ (resp. $\mathcal{H}^p(Y, F)$) onto $\mathcal{L}_{2,\text{abs}}^p$ (resp. $\mathcal{L}_{1,\text{abs}}^{p,\perp}$). Then

$$(6.69) \quad \begin{aligned} \mathcal{L}_{2,\text{abs}}^p \cap (\mathcal{L}_{1,\text{abs}}^p \cap \mathcal{L}_{2,\text{abs}}^p)^\perp &= \text{Im}(P_p Q_p) , \\ \mathcal{L}_{1,\text{abs}}^{p,\perp} \cap (\mathcal{L}_{1,\text{abs}}^{p,\perp} \cap \mathcal{L}_{2,\text{abs}}^{p,\perp})^\perp &= \text{Im}(Q_p P_p) . \end{aligned}$$

And we have the obvious identities

$$(6.70) \quad P_p = \frac{1}{2}(1 + S_2^p) , \quad Q_p = \frac{1}{2}(1 - S_1^p) .$$

Then, by Lemma 6.7, (6.62) and (6.68)-(6.70), the torsion of (6.67) is given by

$$(6.71) \quad \det^*(1 - P_p - Q_p + P_p Q_p + Q_p P_p)^{\frac{1}{4}} = \det^*\left(\frac{2 - S_1^p \circ S_2^p - S_2^p \circ S_1^p}{4}\right)^{\frac{1}{4}} .$$

By (6.59) and (6.60), $T_{\mathcal{L}}$ is the alternative product of the torsions of (6.67) for each p . Then (6.71) implies the first equality in (6.65). We turn to prove the second one.

We denote

$$(6.72) \quad I_{p,\text{abs}} = \det^*\left(\frac{2 - S_1^p \circ S_2^p - S_2^p \circ S_1^p}{4}\right)^{\frac{1}{4}} , \quad I_p = \det^*\left(\frac{2 - C_{12}^p - (C_{12}^p)^{-1}}{4}\right)^{\frac{1}{4}} .$$

It is sufficient to show that

$$(6.73) \quad \prod_p I_{p,\text{abs}}^{(-1)^p} = \prod_p I_p^{(-1)^{p,p}} .$$

By (6.64), we have

$$(6.74) \quad I_p = I_{p,\text{abs}} \cdot I_{p+1,\text{abs}} .$$

By (6.74), we have

$$(6.75) \quad \prod_p I_{p,\text{abs}}^{(-1)^p} = \prod_p I_{p,\text{abs}}^{(-1)^{p,p}} \prod_p I_{p,\text{abs}}^{(-1)^{p-1}(p-1)} = \prod_p I_{p,\text{abs}}^{(-1)^{p,p}} \prod_p I_{p+1,\text{abs}}^{(-1)^{p,p}} = \prod_p I_p^{(-1)^{p,p}} ,$$

which gives exactly (6.73). The proof of Proposition 6.8 is completed. \square

Proof of Theorem 0.2. We equip all the objects in (6.16) with metrics. All the metrics mentioned below are defined/recalled in §6.2.

- $\mathcal{H}^\bullet(Z_{12,\infty}, F)$ is equipped with metric $\|\cdot\|_{\mathcal{H}^\bullet(Z_{12,\infty}, F), R}$;
- $\mathcal{H}_{L^2}^\bullet(Z_{12,\infty}, F) \subseteq \mathcal{H}^\bullet(Z_{12,\infty}, F)$ is equipped with the restricted metric ;
- $\mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F)$ ($j = 1, 2$) is equipped with metric $\|\cdot\|_{\mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F), R}$;
- $\mathcal{H}_{L^2}^\bullet(Z_{j,\infty}, F) \subseteq \mathcal{H}_{\text{bd}}^\bullet(Z_{j,\infty}, F)$ is equipped with the restricted metric ;
- $\mathcal{L}_1^\bullet \cap \mathcal{L}_2^\bullet$ is equipped with metric $\|\cdot\|_{\mathcal{L}_1^\bullet \cap \mathcal{L}_2^\bullet}$;
- $\mathcal{L}_{j,\text{bd}}^\bullet$ ($j = 1, 2$) is equipped with metric $\|\cdot\|_{\mathcal{L}_{j,\text{bd}}^\bullet}$.

Let $T_{h,j}$ ($j = 1, 2, 3$) be the torsion of j -th row, $T_{v,j}$ ($j = 1, \dots, 3n+3$) be the torsion of j -th column. By Proposition 6.2, we have

$$(6.76) \quad \mathcal{T}_R = (1 + \mathcal{O}(e^{-cR})) T_{h,2} .$$

By [6, Theorem 1.20], we have

$$(6.77) \quad T_{h,1} T_{h,2}^{-1} T_{h,3} = \prod_{k=1}^{3n+3} T_{v,k}^{(-1)^{k+1}} .$$

By Proposition 6.3, (6.59), (6.60) and (6.62), we have

$$(6.78) \quad \begin{aligned} T_{v,3p+1} &= \left(1 + \mathcal{O}(R^{-1})\right) R^{\frac{1}{2} \dim \mathcal{L}_{1,\text{bd}}^p} , \\ T_{v,3p+2} &= \left(1 + \mathcal{O}(R^{-1})\right) (2R)^{\frac{1}{2} \dim \mathcal{L}_1^p \cap \mathcal{L}_2^p} , \\ T_{v,3p+3} &= \left(1 + \mathcal{O}(R^{-1})\right) R^{\frac{1}{2} \dim \mathcal{L}_{2,\text{bd}}^p} . \end{aligned}$$

By (6.59), (6.60), (6.62) and the fact that the first row in (6.16) consists of canonical injection/projection maps, we have

$$(6.79) \quad T_{h,1} = 1 .$$

We recall that a_p, b_p and d_p are defined in (5.22). By Proposition 6.5, (6.59), (6.60) and (6.62), we have

$$(6.80) \quad T_{h,3} = \left(1 + \mathcal{O}(R^{-1})\right) \left(\prod_{p=1}^n 2^{(-1)^p a_p}\right) \left(\prod_{p=1}^n R^{(-1)^p d_p}\right) T_{\mathcal{L}} .$$

By the exactness of (5.11), we have

$$(6.81) \quad \sum_{p=1}^n (-1)^p \left(\dim \mathcal{L}_{1,\text{bd}}^p - \dim \mathcal{L}_1^p \cap \mathcal{L}_2^p + \dim \mathcal{L}_{2,\text{bd}}^p \right) = 0 ,$$

$$(6.82) \quad \dim \mathcal{L}_1^p \cap \mathcal{L}_2^p = \dim \ker(\beta_{p,\mathcal{L}}) + \dim \text{Im}(\beta_{p,\mathcal{L}}) = a_p + b_p .$$

By (6.76) - (6.82), we get

$$(6.83) \quad \mathcal{T}_R = \left(1 + \mathcal{O}(R^{-1})\right) \left(\prod_{p=1}^n 2^{(-1)^p (a_p - b_p)/2}\right) \left(\prod_{p=1}^n R^{(-1)^p d_p}\right) T_{\mathcal{L}} .$$

By Lemma 5.2, Proposition 6.8 and (6.83), the proof of Theorem 0.2 is completed. \square

7. GLUING FORMULA OF ANALYTIC TORSION

In this section, as an application of our asymptotic analysis on the ζ -determinant and on the Mayer-Vietoris sequence, we prove Theorem 0.3.

In §7.1, we review the Ray-Singer metric and the anomaly formula. In §7.2, applying Theorem 0.1, 0.2, we prove Theorem 0.3, the gluing formula for analytic torsion.

7.1. Ray-Singer metric and Anomaly formula. Let X be a compact manifold (with or without boundary). Let (F, ∇^F) be a flat complex vector bundle over X .

We equip X with a Riemannian metric g^{TX} . We equip F with a Hermitian metric h^F . We suppose that g^{TX} and h^F have a product structure near ∂X (cf. (0.1)).

We pose absolute/relative boundary condition on ∂X . We recall that $H_{\text{bd}}^\bullet(X, F)$ is defined by (1.4), and $\det H_{\text{bd}}^\bullet(X, F)$ is the determinant of $H_{\text{bd}}^\bullet(X, F)$, defined by (0.10).

We recall that $\Omega_{\text{bd}}^\bullet(X, F)$ is defined by (1.5). Let $D_{X, \text{bd}}^F$ be Hodge-de Rham operator acting on $\Omega_{\text{bd}}^\bullet(X, F)$, defined by (0.2). Let $\|\cdot\|_{\det H_{\text{bd}}^\bullet(X, F)}$ be the L^2 -metric on $\det H_{\text{bd}}^\bullet(X, F)$ induced by Hodge Theorem (cf. Theorem 1.1). Let $\zeta(s)$ be the ζ -function of $D_X^{F,2}$ defined by (0.5).

Definition 7.1. The Ray-Singer metric on $\det H_{\text{bd}}^\bullet(X, F)$ is defined as follows,

$$(7.1) \quad \|\cdot\|_{\det H_{\text{bd}}^\bullet(X, F)}^{\text{RS}} = \|\cdot\|_{\det H_{\text{bd}}^\bullet(X, F)} \exp\left(\frac{1}{2}\zeta'(0)\right).$$

Let $g^{TX'}$ be another Riemannian metric on X . We suppose that g^{TX} and $g^{TX'}$ coincide on a neighborhood of ∂X . Let $\|\cdot\|_{\det H_{\text{bd}}^\bullet(X, F)}^{\text{RS}'}$ be the Ray-Singer metric associated to $g^{TX'}$ and h^F . Before stating the anomaly formula calculating the ratio of $\|\cdot\|_{\det H_{\text{bd}}^\bullet(X, F)}^{\text{RS}}$ and $\|\cdot\|_{\det H_{\text{bd}}^\bullet(X, F)}^{\text{RS}'}$, we define the Euler form and its Chern-Simons form.

Let $o(TX)$ be the orientation bundle of TX . Let ∇^{TX} be the Levi-Civita connection on TX with curvature $R^{TX} = (\nabla^{TX})^2$. We define its Euler form (cf. [10, (4.9)])

$$(7.2) \quad e(TX, \nabla^{TX}) = \text{Pf}\left[\frac{R^{TX}}{2\pi}\right] \in \Omega^{\dim X}(X, o(TX)).$$

Now, let $(g_s^{TX})_{s \in [0,1]}$ be a smooth family of Riemannian metrics on TX such that $g_0^{TX} = g^{TX}$, $g_1^{TX} = g^{TX'}$ and all the g_s^{TX} coincide on a neighborhood of ∂X . Let ∇_s^{TX} be the Levi-Civita connection associated to g_s^{TX} . Set

$$(7.3) \quad \begin{aligned} & \tilde{e}\left(TX, (\nabla_s^{TX})_{s \in [0,1]}\right) \\ &= \int_0^1 \left\{ \frac{\partial}{\partial b} \Big|_{b=0} \text{Pf}\left[\frac{1}{2\pi} (\nabla_s^{TX})^2 + \frac{b}{2\pi} \left(\frac{\partial}{\partial s} \nabla_s^{TX} - \frac{1}{2} \left[\nabla_s^{TX}, (g_s^{TX})^{-1} \frac{\partial}{\partial s} g_s^{TX} \right] \right) \right] \right\} ds. \end{aligned}$$

Same as [10, (4.10)], we have

$$(7.4) \quad d\tilde{e}\left(TX, (\nabla_s^{TX})_{s \in [0,1]}\right) = e(TX, \nabla^{TX'}) - e(TX, \nabla^{TX}).$$

We are in a special case of [12, Theorem 1.9] : since g_s^{TX} coincide near ∂X , the boundary term \tilde{e}_b in [12, (1.45)] vanishes, then the image of $\tilde{e}\left(TX, (\nabla_{s'}^{TX})_{s' \in [0,1]}\right)$ in

$$(7.5) \quad \Omega^{\dim X - 1}(X, o(TX)) / \left\{ d\alpha : \alpha \in \Omega^{\dim X - 2}(X, o(TX)) , \text{supp}(\alpha) \cap \partial X = \emptyset \right\} ,$$

denoted by $\tilde{e}\left(TX, \nabla^{TX}, \nabla^{TX'}\right)$, is independent to the choice of path $(\nabla_s^{TX})_{s \in [0,1]}$, which may be identified to the secondary Euler class in [12, Theorem 1.9].

We define

$$(7.6) \quad \theta(F, h^F) = \text{Tr} \left[(h^F)^{-1} \nabla^F h^F \right] \in \Omega^1(X) ,$$

which is closed (cf. [10, Proposition 4.6]).

The following theorem is a restricted version of the anomaly formula for manifolds with boundary [12, Theorem 0.1], which extends the anomaly formula for closed manifolds [10, Theorem 0.1].

Theorem 7.2. *We have*

$$(7.7) \quad \log \left(\frac{\| \cdot \|_{\det H_{\text{bd}}^{\bullet}(X,F)}^{\text{RS}'}}{\| \cdot \|_{\det H_{\text{bd}}^{\bullet}(X,F)}^{\text{RS}}} \right)^2 = - \int_X \theta(F, h^F) \tilde{e}(TX, \nabla^{TX}, \nabla^{TX'}) .$$

7.2. Gluing formula : proof of Theorem 0.3. We use the notations and assumptions of §3.1. We recall that $\varrho \in \lambda(F)$ is defined by (0.17). In the same way, we define

$$(7.8) \quad \varrho_R \in \lambda_R(F) := \left(\det H^{\bullet}(Z_R, F) \right)^{-1} \otimes \det H_{\text{bd}}^{\bullet}(Z_{1,R}, F) \otimes \det H_{\text{bd}}^{\bullet}(Z_{2,R}, F) .$$

The commutative diagram (6.2) induces an isomorphism $\varphi_{R*} : \lambda_R(F) \rightarrow \lambda(F)$. By the functoriality of the construction of ϱ , we have

$$(7.9) \quad \varphi_{R*} \varrho_R = \varrho .$$

Let $\| \cdot \|_{\det H^{\bullet}(Z_R, F)}^{\text{RS}}$ be the Ray-Singer metric on $\det H^{\bullet}(Z_R, F)$. Let $\| \cdot \|_{\det H_{\text{bd}}^{\bullet}(Z_{j,R}, F)}^{\text{RS}}$ ($j = 1, 2$) be the Ray-Singer metric on $\det H_{\text{bd}}^{\bullet}(Z_{j,R}, F)$. Let $\| \cdot \|_{\lambda_R(F)}^{\text{RS}}$ be the induced metric on $\lambda_R(F)$.

Lemma 7.3. *For any $R > 0$, we have*

$$(7.10) \quad \|\varrho_R\|_{\lambda_R(F)}^{\text{RS}} = \|\varrho\|_{\lambda(F)}^{\text{RS}} .$$

Proof. We use the convention $Z_0 = Z$ and $Z_{0,R} = Z_R$. We identify $H_{\text{bd}}^{\bullet}(Z_{j,R}, F)$ ($j = 0, 1, 2$) to $H_{\text{bd}}^{\bullet}(Z_j, F)$ via φ_{R*} . By (7.8) and (7.9), it is equivalent to show that

$$(7.11) \quad \sum_{j=0}^2 (-1)^{(j-1)(j-2)/2} \log \left(\frac{\| \cdot \|_{\det H_{\text{bd}}^{\bullet}(Z_j, F), R'}^{\text{RS}}}{\| \cdot \|_{\det H_{\text{bd}}^{\bullet}(Z_j, F), R}^{\text{RS}}} \right)^2 = 0 , \quad \text{for any } R' > R \geq 0 .$$

Let $\nabla^{TZ_{j,R}}$ ($j = 1, 2$) be the Levi-Civita connections on $TZ_{j,R}$. We recall that the diffeomorphism $\tilde{\varphi}_{R,R'} : Z_R \rightarrow Z_{R'}$ is constructed in the proof of Proposition 3.3. By restriction, $\tilde{\varphi}_{R,R'}$ induces also the diffeomorphism $\tilde{\varphi}_{R,R'} : Z_{j,R} \rightarrow Z_{j,R'}$ ($j = 1, 2$). We

choose $g_s^{TZ_R} = (1-s)g^{TZ_R} + s\tilde{\varphi}_{R,R'}^* g^{TZ_{R'}}$. Let $g_s^{TZ_{j,R}}$ ($j = 1, 2$) be the restricted metric on $Z_{j,R}$. Let $\nabla_s^{TZ_{j,R}}$ ($j = 0, 1, 2$) be the associated Levi-Civita connections. By (7.3),

$$(7.12) \quad \tilde{e} \left(TZ_R, (\nabla_s^{TZ_R})_{s \in [0,1]} \right) \Big|_{Z_{j,R}} = \tilde{e} \left(TZ_{j,R}, (\nabla_s^{TZ_{j,R}})_{s \in [0,1]} \right), \quad \text{for } j = 1, 2.$$

Since $\tilde{\varphi}_{R,R'}$ preserves the metric near the boundary, by (7.7), we get

$$(7.13) \quad \log \left(\frac{\| \cdot \|_{\det H_{\text{bd}}^{\bullet}(Z_{j,F),R'}}^{\text{RS}}}{\| \cdot \|_{\det H_{\text{bd}}^{\bullet}(Z_{j,F),R}}^{\text{RS}}} \right)^2 = - \int_{Z_{j,R}} \theta(F, h^F) \tilde{e} \left(TZ_{j,R}, (\nabla_s^{TZ_{j,R}})_{s \in [0,1]} \right), \quad \text{for } j = 0, 1, 2.$$

By (7.12) and (7.13), we get (7.11). \square

Proof of Theorem 0.3. Recall that $\zeta_{1,R}(s)$, $\zeta_{2,R}(s)$ and $\zeta_R(s)$ are defined in §0.2, and \mathcal{T}_R is defined in §0.3. By (7.1), it is sufficient to show that

$$(7.14) \quad \mathcal{T}_R \exp \left(\frac{1}{2} \zeta'_{1,R}(0) + \frac{1}{2} \zeta'_{2,R}(0) - \frac{1}{2} \zeta'_R(0) \right) = 2^{-\frac{1}{2}\chi(Y,F)}.$$

By Theorem 0.1, 0.2, the left hand side of (7.14) tends to $2^{-\frac{1}{2}\chi(Y,F)}$ as $R \rightarrow \infty$. Meanwhile, by Lemma 7.3, the left hand side of (7.14) is independent to R . This proves (7.14). \square

8. APPENDIX : MATRIX VALUED HOLOMORPHIC FUNCTIONS

Let $(V, \langle \cdot, \cdot \rangle)$ be a Hermitian vector space of dimension m . Let $\| \cdot \|$ be the norm induced by $\langle \cdot, \cdot \rangle$. Let $D \subseteq \mathbb{C}$ be an open disc centered at 0. Let $C : D \rightarrow \text{End}(V)$ be a holomorphic function such that, for any $z \in D \cap \mathbb{R}$, $C(z)$ is a unitary matrix.

The following theorem is proved in [23, Chapter 2.6, Theorem 6.1].

Theorem 8.1. *There exist real holomorphic functions, i.e., their expansions at 0 are of real coefficients, $\theta_1(z), \dots, \theta_m(z)$ in the neighborhood of 0, such that $e^{i\theta_1(z)}, \dots, e^{i\theta_m(z)}$ give all the eigenvalues of $C(z)$.*

Furthermore, there exist $P_1(z), \dots, P_m(z) \in \text{End}(V)$, which are defined for z in the neighborhood of 0 and holomorphic on z , such that $P_j(z)$ is the orthogonal projection to the eigenspace associated to $\theta_j(z)$, i.e.,

$$(8.1) \quad \begin{aligned} 1 &= P_1(z) + \dots + P_m(z), \quad P_j(z)P_k(z) = 0, \quad \text{for } 1 \leq j, k \leq m, \quad j \neq k, \\ C(z) &= e^{i\theta_1(z)}P_1(z) + \dots + e^{i\theta_m(z)}P_m(z). \end{aligned}$$

In the sequel, by shrinking D to a smaller disc if necessary, we suppose that θ_j and P_j ($j = 1, \dots, m$) are all-well defined in the neighborhood of \overline{D} .

For any $R > 0$, we consider the equation

$$(8.2) \quad e^{4iRz}C(z)v = v,$$

where $z \in D$, $v \in V$. By Theorem 8.1, for R and z fixed, (8.2) as an equation of v has non trivial solution if and only if one of $4Rz + \theta_1(z), \dots, 4Rz + \theta_m(z)$ is in $2\pi\mathbb{Z}$.

Proposition 8.2. *There exist $R_0 > 0$, $\varepsilon > 0$, such that, for $R > R_0$, $z_0 \in]-\varepsilon, \varepsilon[$, $v \in V$, if*

$$(8.3) \quad \|e^{4iRz_0}C(z_0)v - v\| < \|v\|,$$

then, there exist $z_1, \dots, z_m \in \mathbb{R}$, $w_1, \dots, w_m \in V$, satisfying

$$(8.4) \quad \begin{aligned} |z_j - z_0|^2 &< \|v\|^{-1} \cdot \|e^{4iRz}C(z_0)v - v\|, \\ \|P_j(z_0)v - w_j\|^2 &< \|v\| \cdot \|e^{4iRz}C(z_0)v - v\|, \\ e^{4iRz_j}C(z_j)w_j - w_j &= 0, \end{aligned}$$

for $j = 1, \dots, m$.

Proof. We equip $\text{End}(V)$ with the operator norm.

We fix $B_1, B_2 > 0$ such that, for any $s, t \in \overline{D}$ and $j = 1, \dots, m$,

$$(8.5) \quad |\theta_j(s) - \theta_j(t)| < B_1 |s - t|, \quad \|P_j(s) - P_j(t)\| < B_2 |s - t|.$$

We choose $\varepsilon > 0$, $R_0 > 0$ such that

$$(8.6) \quad \left[-\varepsilon - \frac{2\pi}{4R_0 - B_1}, \varepsilon + \frac{2\pi}{4R_0 - B_1} \right] \subseteq D, \quad 0 < \frac{2}{4R_0 - B_1} < 1, \quad 0 < \frac{2B_2}{4R_0 - B_1} < 1.$$

Set $v_j = P_j(z_0)v$. Then, for $R > R_0$, by (8.1),

$$(8.7) \quad e^{4iRz_0}C(z_0)v - v = \sum_{j=1}^m (e^{4iRz_0+i\theta_j(z_0)} - 1) v_j,$$

Since these v_j are mutually orthogonal, we have

$$(8.8) \quad |e^{4iRz_0+i\theta_j(z_0)} - 1| \cdot \|v_j\| \leq \|e^{4iRz_0}C(z_0)v - v\|.$$

If $\|v_j\|^2 < \|v\| \cdot \|e^{4iRz}C(z_0)v - v\|$, set $w_j = 0$, $z_j = z_0$. In this case, (8.4) holds trivially. Otherwise, by (8.3) and (8.8), we have

$$(8.9) \quad |e^{4iRz_0+i\theta_j(z_0)} - 1|^2 \leq \|v\|^{-1} \cdot \|e^{4iRz_0}C(z_0)v - v\| < 1.$$

Then, there exists $k_j \in \mathbb{Z}$, such that,

$$(8.10) \quad |4Rz_0 + \theta_j(z_0) - 2k_j\pi|^2 \leq 4\|v\|^{-1} \cdot \|e^{4iRz_0}C(z_0)v - v\|.$$

For $R > R_0$, by (8.5) and (8.6), $4Rz + \theta_j(z) - 2k_j\pi$ as a function of $z \in \mathbb{R}$ is strictly increasing, and its derivative is greater than $4R - B_1$. Let $z_j \in \mathbb{R}$ be the unique real number satisfying $4Rz_j + \theta_j(z_j) - 2k_j\pi = 0$, then

$$(8.11) \quad |z_j - z|^2 < \left(\frac{2}{4R - B_1} \right)^2 \|v\|^{-1} \cdot \|e^{4iRz_0}C(z_0)v - v\|.$$

By (8.6) and (8.11), the first equation of (8.4) holds. Set $w_j = P(z_j)v$, then the third equation of (8.4) holds trivially. Furthermore, by the choice of B_2 , we have

$$(8.12) \quad \|P_j(z_0)v - w_j\| = \|(P_j(z_0) - P_j(z_j))v\| \leq \|P_j(z_0) - P_j(z_j)\| \cdot \|v\| \leq B_2 |z_0 - z_j| \cdot \|v\|.$$

By (8.6), (8.11) and (8.12), the second equation of (8.4) holds. \square

For $R > 0$, set

$$(8.13) \quad \begin{aligned} \Lambda_R(C) &= \left\{ \rho > 0 : \det(e^{4iR\rho}C(\rho) - 1) = 0 \right\}, \\ \Lambda_R^*(C) &= \left\{ \lambda > 0 : \det(e^{4iR\lambda}C(0) - 1) = 0 \right\}. \end{aligned}$$

We fix $\kappa > 0$.

Proposition 8.3. *There exist $a > 0$, $R_0 > 0$, such that, for any $R > R_0$, $R^{-1+\kappa} \leq \gamma \leq 1$ and $f \in \mathcal{C}^1(\mathbb{R})$, we have*

$$(8.14) \quad \left| \sum_{\rho \in \Lambda_R(C), |\rho| < \gamma} f(\rho) - \sum_{\lambda \in \Lambda_R^*(C), |\lambda| < \gamma} f(\lambda) \right| \leq a\gamma^2 \sup_{|x| \leq \gamma} |f'(x)| + a\gamma \sup_{|x| \leq \gamma} |f(x)|.$$

Proof. By Theorem 8.1, we may suppose that $C(\rho) = e^{i\theta(\rho)}$ for certain analytic function θ . The rest of the proof is a direct estimate, and we leave it to readers. \square

Set

$$(8.15) \quad \zeta_{C,R}(s) = - \sum_{\lambda \in \Lambda_R^*(C)} (\lambda^2)^{-s}.$$

We recall that $m = \dim V$. Set $r = \dim \ker (C(0) - 1)$.

Proposition 8.4. *If $\mathrm{Sp}(C(0)) = \overline{\mathrm{Sp}(C(0))}$, then*

$$(8.16) \quad \zeta_{C,R}'(0) = r \log(2R) + m \log 2 + \frac{1}{2} \log \det^* \left(\frac{2 - C(0) - C(0)^{-1}}{4} \right).$$

Proof. As special cases of Hurwitz ζ -functions (cf. [39, §7]), we have

$$(8.17) \quad - \frac{\partial}{\partial s} \Big|_{s=0} \sum_{k=1}^{\infty} \left(\frac{2\pi k - \theta}{4R} \right)^{-2s} = \begin{cases} \log(4R) & \text{for } \theta = 0, \\ \frac{1}{2} \log(2 - 2 \cos \theta) & \text{for } 0 < \theta \leq \pi. \end{cases}$$

Since $C(0)$ is diagonalizable, it suffices to consider the following cases.

Case 1. $m = 1$, $r = 1$, $C = 1$, then (8.16) is equivalent to (8.17) with $\theta = 0$.

Case 2. $m = 1$, $r = 0$, $C = -1$, then (8.16) is equivalent to (8.17) with $\theta = \pi$.

Case 3. $m = 2$, $r = 0$, $\mathrm{Sp} C = \{e^{i\alpha}, e^{-i\alpha}\}$ with $\alpha \in]0, \pi[$, then (8.16) is equivalent to (8.17) with $\theta = \alpha$. \square

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